On the development of mathematical thought ...

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CHAPTER XIII.

ON THE DEVELOPMENT OF MATHEMATICAL THOUGHT DURING THE NINETEENTH CENTURY.

In venturing upon the last and most abstract portion of the great domain of Scientific Thought of the century, it may be well to remind the reader that it is not a history of science but a history of thought that I am writing. When dealing in the foregoing chapters with manifold discoveries, drawn promiscuously from the various natural sciences, I have done so only to show how the scientific mind has, in the course of the period, come to regard the things of nature from different points of view, and to think and reason on them differently. Such changes have frequently been brought about by the discovery of novel facts, but this alone has not generally sufficed to mark also a change in the manner of reasoning on and thinking about them. The increase in the number of natural species, of the chemical elements or of the smaller planets, has not necessarily made us think differently about these things in themselves: the theory and point of view may change without any change in the object towards which they are directed, 1. History of thought.

for they mark more the attitude of the beholder than the things which he regards. It is true that a very small addition to our actual knowledge of facts, like the sudden appearance of some characteristic feature in a landscape, may sometimes entirely alter the whole aspect, induce us to abandon our accustomed views, and call up suddenly an unforeseen train of ideas; in such a case, perhaps, this insignificant discovery becomes historically interesting, although it is mainly by the altered trains of thought which it has evoked that it has become important to us.

2. Difference between thought and knowledge.

The difference of scientific knowledge and scientific thought is thus owing to the two factors which are involved—the facts of science or nature on the one side and the scientifically thinking mind on the other. Now it might appear as if this difference vanished when we approach the abstract science of mathematics, or at least that of number; for in numbering and counting we have really only to do with a process of thought, and it would seem as if the science of number were itself the science of thought, or at least a portion of it. In fact, the question arises, Is there any difference between mathematical science and mathematical thought? Some considerations might induce us to think that there is not. On the other side, I shall try to show in this chapter that there is, and that the development of mathematics during our period has brought this out very clearly and prominently.

3. Popular prejudices regarding mathematics.

There is an opinion current among many thinking persons who have not occupied themselves with mathematical science, though they may be very efficient in

calculating and measuring, that there is really nothing new in mathematics, that two and two always make four, that the sum of the angles in a triangle always make two right angles, and that all progress in mathematics is merely a question of intricacy, a never-ending process of increased complication by which you can puzzle even the cleverest calculator. To them the history of mathematics would be something analogous to the history of games like whist or chess, the resources and complications of which seem to be inexhaustible. So they think[1] that the intricacies and refinements of elementary and higher mathematics will supply endless material for training the minds of schoolboys or trying the ingenuity

[1] "Some people have been found to regard all mathematics, after the 47th proposition of Euclid, as a sort of morbid secretion, to be compared only with the pearl said to be generated in the diseased oyster, or, as I have heard it described, 'une excroissance maladive de l'esprit humain.' Others find its justification, its *raison d'être*, in its being either the torch-bearer leading the way, or the handmaiden holding up the train of Physical Science; and a very clever writer in a recent magazine article expresses his doubts whether it is, in itself, a more serious pursuit, or more worthy of interesting an intellectual human being, than the study of chess problems or Chinese puzzles. What is it to us, they say, if the three angles of a triangle are equal to two right angles, or if every even number is, or may be, the sum of two primes, or if every equation of an odd degree must have a real root? How dull, stale, flat, and unprofitable are such and such like announcements! Much more interesting to read an account of a marriage in high life, or the details of an international boat-race. But this is like judging of architecture from being shown some bricks and mortar, or even a quarried stone of a public building, or of painting from the colours mixed on the palette, or of music by listening to the thin and screech sounds produced by a bow passed haphazard over the strings of a violin. The world of ideas which it discloses or illuminates, the contemplation of divine beauty and order which it induces, the harmonious connexion of its parts, the infinite hierarchy and absolute evidence of the truths with which it is concerned, these, and such like, are the surest grounds of the title of mathematics to human regard, and would remain unimpeached and unimpaired were the plan of the universe unrolled like a map at our feet, and the mind of man qualified to take in the whole scheme of creation at a glance" (Prof. J. J. Sylvester, Address before Brit. Assoc., see 'Report,' 1869, p. 7).

of senate-house examiners and examinees, without for a moment considering the question whether mathematical thought as distinguished from mathematical problems is capable of and has undergone any radical and fundamental change or development.

4. Use of mathematics.

Closely allied with this is the further question as to the use of mathematics. Two extreme views have always existed on this point.[1] To some, mathematics is only a measuring and calculating instrument,[2] and their interest

[1] Of the two greatest mathematicians of modern times, Newton and Gauss, the former can be considered as a representative of the first, the latter of the second class; neither of them was exclusively so, and Newton's inventions in the pure science of mathematics were probably equal to Gauss's work in applied mathematics. Newton's reluctance to publish the method of fluxions invented and used by him may perhaps be attributed to the fact that he was not satisfied with the logical foundations of the calculus; and Gauss is known to have abandoned his electro-dynamic speculations, as he could not find a satisfactory physical basis (see *supra*, p. 67). Others who were not troubled by similar logical or practical scruples stepped in and did the work, to the great benefit of scientific progress. Newton's greatest work, the 'Principia,' laid the foundation of mathematical physics; Gauss's greatest work, the 'Disquisitiones Arithmeticæ,' that of higher arithmetic as distinguished from algebra. Both works, written in the synthetic style of the ancients, are difficult, if not deterrent, in their form, neither of them leading the reader by easy steps to the results. It took twenty or more years before either of these works received due recognition; neither found favour at once before that great tribunal of mathematical thought, the Paris Academy of Sciences. Newton's early reputation was established by other researches and inventions, notably in optics; Gauss became known through his theoretical rediscovery of Ceres, the first of the minor planets (see above, vol. i. p. 182). The country of Newton is still preeminent for its culture of mathematical physics, that of Gauss for the most abstract work in mathematics. Not to speak of living authorities, I need only mention Stokes and Clerk-Maxwell on the one side, Grassmann, Weierstrass, and Georg Cantor on the other.

[2] Huxley said: "Mathematics may be compared to a mill of exquisite workmanship which grinds you stuff of any degree of fineness: but, nevertheless, what you get out depends on what you put in; and as the grandest mill in the world will not extract wheat-flour from peas-cods, so pages of formulæ will not get a definite result out of loose data"; and on another occasion he said that mathematics "is that study which knows nothing of observation, nothing of induction, nothing of experiment, nothing of causation." The former statement was endorsed by Lord Kelvin ('Pop. Lectures,' &c., vol. ii. p.

ceases as soon as discussions arise which cannot benefit those who use the instrument for the purposes of application in mechanics, astronomy, physics, statistics, and other sciences. At the other extreme we have those who are animated exclusively by the love of pure science. To them pure mathematics, with the theory of numbers[1] at the head, is the one real and genuine science, and the applications have only an interest in so far as they contain or suggest problems in pure mathematics. They are mainly occupied with examining and strengthening the foundations of mathematical reasoning and purifying its methods, inventing rigorous proofs, and testing the validity and range of applicability of current conceptions. We may say that the former are led by practical, the latter by philosophical, interests, and these latter may be either logical or ontological,[2]

102); the latter was energetically repudiated by Sylvester in his famous Address to the first section of the British Assoc. at Exeter (1869, 'Report,' &c., p. 1, &c.)

[1] Gauss considered mathematics to be "the Queen of the Sciences, and arithmetic the Queen of Mathematics. She frequently condescends to do service for astronomy and other natural sciences, but to her belongs, under all circumstances, the foremost place" (see 'Gauss zum Gedächtniss,' by Sartorius von Waltershausen, Leipzig, 1856, p. 79). Cayley's presidential Address to the British Association, 1883, has been frequently quoted: "Mathematics connect themselves on one side with common life and the physical sciences; on the other side with philosophy in regard to our notions of space and time and the questions which have arisen as to the universality and necessity of the truths of mathematics, and the foundation of our knowledge of them. I would remark here that the connection (if it exists) of arithmetic and algebra with the notion of time is far less obvious than that of geometry with the notion of space" ('Mathematical Papers,' vol. xi. p. 130). In addition to founding higher arithmetic, Gauss occupied himself with the foundations of geometry, and, as he expected much from the development of the theory of numbers, so he placed "great hopes on the cultivation of the *geometria situs*, in which he saw large undeveloped tracts which could not be conquered by the existing calculus" (Sartorius, *loc. cit.*, p. 88).

[2] To this might be added the psychological interest which attaches to mathematical conceptions. The late Prof. Paul Du Bois-Reymond occupied himself

inasmuch as number and form are considered to be the highest categories of human thought, or likewise as the ultimate elements of all reality. These two interests existed already in antiquity,[1] as the word "geometry"

5. Twofold interest in mathematics.

much with the question. See the following works: 'Die Allgemeine Functionentheorie,' part i., Tübingen, 1882; 'Ueber die Grundlagen der Erkenntniss in den exacten Wissenschaften,' Tübingen, 1890; and his paper "Ueber die Paradoxien des Infinitärcalcüls" ('Mathematische Annalen,' vol. ix. p. 149). In addition to the two main interests which attach to mathematical research, and which I distinguish as the practical and the philosophical, a third point of view has sprung up in modern times which can be called the purely logical. It proposes to treat any special development of mathematical research with the aid of a definite, logically connected complex of ideas, and not to be satisfied to solve definite problems with the help of any methods which may casually present themselves, however ingenious they may be. In this way the great geometrician, Jacob Steiner, *e.g.*, refused the assistance of analysis in the solution of geometrical problems, conceiving geometry as a complete organism which should solve its problems by its own means. This view has been much strengthened by the development in modern times of the theory of Groups; a group of operations being defined as a sequence of such operations as always lead back again to operations of the same kind. Mathematical rigorists in this sense would look upon the use of mixed methods or operations not belonging to the same group with that kind of disfavour with which we should regard an essayist who could not express his ideas in pure English, but was obliged to import foreign words and expressions. It is interesting to see that the country which has offended most by the importation of foreign words — namely, Germany—is that in which this purism in mathematical taste has found the most definite expression. (See, *inter alia*, Prof. Friedrich Engel's Inaugural Lecture, "Der Geschmack in der neueren Mathematik," Leipzig, 1890, as also Prof. F. Klein's suggestive tract, 'Vergleichende Betrachtungen über neuere Geometrische Forschungen,' Erlangen, 1872.)

[1] The literature of this subject is considerable. I confine myself to two works. The late eminent mathematician, Hermann Hankel, of whom more in the sequel of this chapter, besides showing much originality in the higher branches of the science, took great interest in its philosophical foundations and historical beginnings. In 1870 he published a small but highly interesting volume, 'Zur Geschichte der Mathematik in Alterthum und Mittelalter' (Leipzig, Teubner). We have, besides, the great work of Prof. Moritz Cantor, 'Vorlesungen über Geschichte der Mathematik,' in three large volumes (Leipzig, Teubner). It brings the history down to 1758. Referring to the two interests which led to mathematical investigations, Hankel says (p. 88): "From the moment that Greek philosophers begin to attract our attention through their mathematical researches, the aspect which mathematics present

and the well-known references to mathematical ideas in the schools of Pythagoras and Plato indicate. An ancient fragment[1] which enumerates briefly the Grecian mathematicians, says of Pythagoras, "He changed the occupation with this branch of knowledge into a real science, inasmuch as he contemplated its foundation from a higher point of view, and investigated the theorems less materially and more intellectually;[2] and of Plato it says that "He filled his writings with mathematical discussions, showing everywhere how much geometry attaches itself to philosophy."[3]

This twofold connection of mathematical with other pursuits has, after the lapse of many centuries, come prominently forward again in the nineteenth century. We have already had to record a powerful stimulus to mathematical thought in almost every chapter in which we dealt with the fruitful ideas which governed scientific work, and we have now no less to draw attention to the philosophical treatment which has been bestowed upon the foundations of science and the inroad of mathemati-

changes radically. Whilst among the earlier civilised nations we only meet with routine and practice, with empirical rules which served practical purposes in an isolated manner, the Grecian mind on the other side recognised, from the first moment when it became acquainted with this matter, that it contained something which transcended all those practical ends, but which was worthy of special attention, and which could be expressed in a general form, being, in fact, an object of science. This is the high merit of the Greek mathematicians; nor need one fear that this merit should be diminished by admitting that they borrowed the new material from the ancient Egyptian civilisation."

[1] The fragment referred to is preserved by Proclus, and is given in full in Cantor's work (vol. i. p. 124 *sqq.*) He calls it an ancient catalogue of mathematicians. It is generally attributed to Eudemus of Rhodes, who belonged to the peripatetic school of Philosophy, and was the author of several historical treatises on geometry and astronomy (Cantor, vol. i. p. 108).

[2] Cantor, vol. i. p. 137.

[3] Ibid., p. 213.

cal into philosophical thought;[1] so much so that this closing chapter on the development of mathematical thought forms a fitting link with the next great department of our subject—the Philosophy of the Century.

6. Origin of mathematics.

We are told that mathematics among the Greeks had its origin in the Geometry invented by the ancient Egyptians for practical surveying purposes. The first mathematical problems arose in the practice of mensuration. Modern mathematical thought received in an analogous manner its greatest stimulus through the Uranometry of Kepler, Newton, and Laplace: through the mechanics and the survey of the heavens new methods for solving astronomical problems were invented in the seventeenth and eighteenth centuries, and the nineteenth century can be said to have attempted to perform towards this new body of doctrine the same task that Euclid, three hundred years before the Christian era, performed towards the then existing mathematics. As Proclus tells us, "putting together the elements, arranging much from Eudoxus, furnishing much from Theætetus, he, moreover, subjected to rigorous proofs what had been negligently demonstrated by his predecessors."[2] What one man, so far as we know, did for the Grecian science, a number of great thinkers in

[1] Thus, for instance, the recent investigations and theories of the "manifold," as they have been set forth by Prof. Georg Cantor of Halle, constitute, as it were, a new chapter in mathematical science, whereas they were formerly a subject merely of philosophical interest. See a remark to this effect by B. Kerry at the end of his very interesting article on Cantor's doctrine in the 9th vol. of Avenarius's 'Zeitschrift für wissenschaftliche Philosophie' (1885), p. 231, where he refers to Kant's comparison of philosophy to a Hecuba "tot generis natisque potens."

[2] Quoted by Cantor, vol. i. p. 247. See also Hankel, *loc. cit.*, p. 381 *sqq.*

our century, among whom I only mention Gauss, Cauchy, and Weierstrass, attempted to do for the new science which was created during the two preceding centuries. As Prof. Klein says, "We are living in a critical period, similar to that of Euclid."[1]

[1] See 'The Evanston Colloquium, Lectures on Mathematics delivered in August and September 1893,' by Felix Klein, notably Lecture vi. In this lecture Prof. Klein explains his view (to which he had given utterance in his address before the Congress of Mathematics at Chicago: 'Papers published by the American Mathematical Society,' vol. i. p. 133. New York, 1896) on the relation of pure mathematics to applied science. This view is based upon the distinction between what he calls the "naïve and the refined intuition." . . . "It is the latter that we find in Euclid; he carefully develops his system on the basis of well-formulated axioms, is fully conscious of the necessity of exact proofs, clearly distinguishes between the commensurable and the incommensurable, and so forth. . . . The naïve intuition, on the other hand, was especially active during the period of the genesis of the differential and integral calculus. Thus we see that Newton assumes without hesitation the existence, in every case, of a velocity in a moving point, without troubling himself with the inquiry whether there might not be continuous functions having no derivative."

In the opinion of Prof. Klein "the root of the matter lies in the fact that the naïve intuition is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact."

In the sequel Prof. Klein shows that the naïve intuition imports into the elementary conceptions elements which are left out in the purely logical development, and that this again leads to conclusions which are not capable of being verified by intuition, no mental image being possible. Thus, for instance, the abstract geometry of Lobatchevsky and Riemann led Beltrami to the logical conception of the pseudosphere of which we cannot form any mental image. Similar views to those of Prof. Klein have been latterly expressed by H. Poincaré in his suggestive volume 'La Science et l'Hypothèse' (Paris, 1893). He there says (p. 90): ". . . L'expérience joue un rôle indispensable dans la genèse de la géométrie; mais ce serait une erreur d'en conclure que la géométrie est une science expérimentale, même en partie. . . . La géométrie ne serait que l'étude des mouvements des solides; mais elle ne s'occupe pas en réalité des solides naturels, elle a pour objet certains solides idéaux, absolument invariables, qui n'en sont qu'une image simplifiée et bien lointaine. . . . Ce qui est l'objet de la géométrie c'est l'étude d'un 'groupe' particulier; mais le concept général de groupe préexiste dans notre esprit au moins en puissance. . . . Seulement, parmi tous les groupes possibles, il faut choisir celui qui sera pour ainsi dire l'étalon auquel nous rapporterons les phénomènes naturels." This distinction between the mathematics of intuition and the mathematics of logic has also been forced upon us from quite a different quarter. The complica-

7. Gauss. It is right to place the name of Gauss at the head, for his investigations regarding several fundamental and critical questions in arithmetic and geometry date from the last years of the eighteenth century, long before Cauchy's influence made itself felt. This is now abundantly clear through the publication of Gauss's works, and from much of his correspondence with personal friends, notably with the astronomer Bessel. We can now understand how those who knew him regarded him as a kind of mathematical oracle to whom "nothing in theory existed that he had not looked at from all sides,"[1] and who anticipated in his own mind the development which mathematical thought was to take for a long time after him. And yet it was not to him primarily that the great change was due which came over mathematical reasoning during the first half of the century. Gauss was not a great teacher. In fact, there existed in the first quarter of the period only one great training school in advanced mathematics, and that was Paris.

8. Cauchy. There it was that Augustin Cauchy—first as lecturer,

tion of modern mathematics and the refinement of the modern theories have brought about the desire "to create an abridged system of mathematics adapted to the needs of the applied sciences, without passing through the whole realm of abstract mathematics" (Klein, *loc. cit.*, p. 48). In this country Prof. Perry has made a beginning by publishing his well-known work, 'Calculus for Engineers,' which has been welcomed by Prof. Klein in Germany, and which has led to an extensive correspondence in the pages of 'Nature'; it being recognised by many that a quicker road must be made from the elements to the higher applications of mathematics in the natural sciences than the present school system, beginning with Euclid, admits of. The separation of the logical and practical treatment of any science, as likewise the independent development in Germany of the polytechnic school alongside of the university, has, however, its dangers, as is recognised by Prof. Klein ('Chicago Mathematical Papers,' p. 136).

[1] See Bessel's letter to Gauss 27th December 1810, in 'Briefwechsel zwischen G. and B.,' Leipzig, 1880, p. 132.

then as professor—exerted his great influence in the famous École Polytechnique, in the Sorbonne, in the Collége de France.[1] In contrast with Gauss—who was self-contained, proud, and unapproachable, whose finished and perfect mathematical tracts were, even to those who worshipped him, an abomination,[2] owing to their unintelligible and novel enunciation, who hated lecturing—Cauchy possessed the enthusiasm and patience of the teacher,[3] spent hours with his pupils, and published his lectures on the foundations of the Calculus for the benefit of the rising mathematical generation. Thus he has the merit of having created a new school of mathematical thought—not only in France but also abroad, where the greatest intellects, such as that of Abel,[4] expressed themselves indebted to him for having pointed out the only right road of progress. It will be useful to define somewhat more closely wherein this new school differed from that preceding it, which culminated in the great names of Euler, Lagrange, and Laplace.

The great development of modern as compared with ancient mathematics may be stated as consisting in the in-

[1] See Valson, 'La Vie et les Travaux du Baron Cauchy,' Paris, 1868, vol. i. p. 60 *sqq.*

[2] "On disait que sa manière d'exposer était mauvaise, ou encore qu'il faisait comme le renard, qui efface avec sa queue les traces de ses pas sur le sable. Crelle dit, selon Abel, que tout ce qu'écrit Gauss n'est qu'abomination (Gräuel), car c'est si obscur qu'il est presque impossible d'y rien comprendre" (Bjerknes, 'Niels Henrik Abel,' Trad. française, Paris, 1885, p. 92).

[3] "C'est que Cauchy alliait au génie des Euler, des Lagrange, des Laplace, des Gauss, des Jacobi, l'amour de l'enseignement porté jusqu'à l'enthousiasme, une rare bonté, une simplicite, une chaleur de cœur qu'il a conservées jusqu'à la fin de sa vie" (Combes, quoted by Valson, vol. i. p. 63).

[4] See Bjerknes, 'N.-H. Abel,' p. 48 *sqq.*; p. 300. Cauchy's 'Cours d'Analyse' appeared in 1821; the 'Résumé des leçons sur le calcul infinitésimal,' to which Abel refers in a letter to Holmboe, dated 1826, appeared in 1823.

troduction of algebra or general arithmetic, in the application of this to geometry and dynamics, and in the invention of the infinitesimal methods, through which the rigorous theorems of the older geometricians which referred to the simpler figures—such as straight lines, circles, spheres, cones, &c.—became applicable to the infinite variety of curves and surfaces in which the objects and phenomena of nature present themselves to our observation. Logically speaking, it was a grand process of generalisation, based mostly on inference and induction, sometimes merely on intuition.[1] Such a process of generalisation has a twofold effect on the progress of science.

9. Process of generalisation.

The first and more prominent result was the greatly increased power of dealing with special problems which the generalised method affords, and the largely increased field of research which it opened out. We may say that the century which followed the inventions of Descartes, Newton, and Leibniz, was mainly occupied in exploring the new field which had been disclosed, in formulating and solving the numberless problems which presented themselves on all sides; also, where complete and rigorous solutions seemed unattainable, in inventing methods of approximation which were useful for practical purposes. In this direction so much had to be done, so much work lay ready to hand, that the second and apparently less practical effect of the new generalisations receded for a time into the background. We may term

[1] "On se reportait inconsciemment au modèle qui nous est fourni par les fonctions considérées en mécanique et on rejetait tout ce qui s'écartait de ce modèle; on n'était pas guidé par une définition claire et rigoureuse, mais par une sorte d'intuition et d'obscur instinct" (Poincaré, "L'œuvre math. de Weierstrass," 'Acta Mathematica,' vol. xxii. p. 4).

this second and more hidden line of research the logical side of the new development. It corresponds to the work which Euclid performed in ancient geometry, the framing of clear definitions and of unambiguous axioms; proceeding from these by rigorous reasoning to the theorems of the new science.[1] But the translation of geometrical and mechanical conceptions into those of generalised arithmetic or algebra brought with it a logical problem of quite a novel kind which has given to modern mathematics quite a new aspect. This new problem is the retranslation of algebraical—*i.e.*, of general—formulæ into geometrical conceptions—the geometrical construction of algebraical expressions. It is the inverse operation of the former. In this inversion of any given operation lies the soul and principle of all mathematical progress, both in theory and in application.[2] The invention of

10. Inverse operations.

[1] Referring specially to the definition of a "function" or mathematical dependence, a conception introduced by Euler, but not rigorously defined by him, M. Poincaré says, *loc. cit.*: "Au commencement du siècle, l'idée de fonction était une notion à la fois trop restreinte et trop vague. . . . Cette définition, il fallait la donner: car l'analyse ne pouvait qu'à ce prix acquérir la parfaite rigueur." In its generality this task was performed in the last third of the century by Weierstrass, but the necessity of this criticism of the formulæ invented by modern mathematics dates from the appearance of Cauchy's 'Mémoire sur la théorie des intégrales définies' of 1814, which Legendre reported on in this sense, but which was not published till 1825.

[2] The operations referred to are generally of two kinds: first, there is the operation of translating geometrical relations, intuitively given, into algebraical relations; and, secondly, the operation of extending algebraical relations by going forward or backward in the order of numbers, usually given by indices. In each case the new relations arrived at require to be interpreted, and this interpretation leads nearly always to an extension of knowledge or to novel conceptions. A simple example of the first kind presents itself in the geometrical construction of the higher powers of quantity. Having agreed to define by a the length of a line, by a^2 an area, what is the meaning of a^3 a^4 . . . a^n? Can any geometrical meaning be attached to these symbols? An example of the

the seventeenth century afforded two grand occasions for such progress, and the creation through it of novel mathematical ideas. The translation of geometrical con-

second class is the following: having defined the symbols

$$\frac{dy}{dx}, \frac{d^2y}{dx^2} \cdots \frac{d^ny}{dx^n},$$

an operation suggests itself in the inverse order, the indices or their reciprocals (inversion) being taken negatively. Can any meaning be attached to these latter symbols? Further, if the operation denoted by going on from one of these symbols to the next is known and feasible, how can the inverse operation be carried out? In the first class of problems we proceed from an intuitively given order to a purely logical order, and have in the sequel to go back from the purely logical order to an intuitive order of ideas. In the second case, having followed a certain logical order, we desire to know what the inversion of this order will produce and how it can be carried out. The view that the direct and indirect processes of thought form the basis of all mathematical reasoning, and an alternation of the two the principle of progress, has been for the first time consistently expounded by Hermann Hankel in his 'Theorie der Complexen Zahlen-Systeme,' Leipzig, 1867. But it had already been insisted on by George Peacock in his "Report," &c., contained in the 3rd vol. of the 'Reports of the Brit. Assoc.,' 1833, where he says (p. 223): "There are two distinct processes in Algebra, the direct and the inverse, presenting generally very different degrees of difficulty. In the first case, we proceed from defined operations, and by various processes of demonstrative reasoning we arrive at results which are general in form though particular in value, and which are subsequently generalised in value likewise; in the second, we commence from the general result, and we are either required to discover from its form and composition some equivalent result, or, if defined operations have produced it, to discover the primitive quantity from which those operations have commenced. Of all these processes we have already given examples, and nearly the whole business of analysis will consist in their discussion and development, under the infinitely varied forms in which they will present themselves."

It is extraordinary how little influence this very interesting, comprehensive, and up-to-date report on Continental mathematics, including the works of Gauss, Cauchy, and Abel, seems to have had on the development of English mathematics. But the latter have through an independent movement —viz., the invention of the Calculus of Operations — led on to the radical change which has taken place in recent mathematical thought. This change, which can be explained by saying that the science of Magnitude must be preceded by the doctrine of Forms or Relations, and that the science of Magnitude is only a special application of the science of Forms, was independently prepared by Hermann Grassmann, of whom Hankel says (*loc. cit.*, p. 16): "The idea of a doctrine of Forms which should precede a doctrine of Magnitude, and of considering the latter from the point of view of the former, . . . remained of

ceptions into algebraical language suggested the inverse operation of interpreting algebraical terms by geometrical conceptions, and led to an enormous extension of geometrical knowledge.[1] Further, the infinitesimal methods through which curves and curved surfaces were conceived as being made up of an infinite number of infinitesimally small, straight—*i.e.*, measurable—lines, led to the inverse problem; given any algebraical operations which obtain only in infinitesimally small dimensions—*i.e.*, at the limit—how do they sum up to finite quantities and

little value for the development of mathematics, so long as it was only used to prove theorems which besides being already known, were sufficiently though merely empirically proved. It was H. Grassmann who took up this idea for the first time in a truly philosophical spirit and treated it from a comprehensive point of view." Hankel also refers to Peacock as well as to De Morgan, whose writings, however, he was insufficiently acquainted with (ibid., p. 15). In quite recent times Mr A. N. Whitehead has conceived "mathematics in the widest signification to be the development of all types of formal, necessary, deductive reasoning," and has given a first instalment of this development in his 'Treatise on Universal Algebra' (vol. i., Cambridge, 1898). See the preface to this work (pp. 6, 7).

[1] A good example of the use of the alternating employment of the intuitive (inductive) and the logical (deductive) methods is to be found in the modern doctrine of curves. The invention of Descartes, by which a curve was represented by an equation, led to the introduction of the conception of the "degree" or "order" of a curve and its geometrical equivalent; whereas the geometrical conception of the tangent to a curve led to the distinction of curves according to their "class," which was not immediately evident from the equation of the curve but which led to other analytical methods of representation where the tangential properties of curves became more evident. A third method of studying curves was introduced by Plücker (1832), who started from "the singularities" which curves present, defined them, and established his well-known equations. A further study of these "singularities" led to the notion of the "genus" or "deficiency" (Cayley) of a curve. The gradual development of these and further ideas relating to curves is concisely given in an article by Cayley on "Curve" in the 6th vol. of the 'Encyclopædia Britannica,' reprinted in Cayley's collected papers, vol. xi. This article furnishes also a good example of the historical treatment of a purely mathematical subject by showing, not so much the progress of mathematical knowledge of special things, as the development of the manner in which such things are looked at —*i.e.*, of mathematical thought.

figures? What are the properties of these finite figures as inferred from the properties of their infinitesimally small parts? The infinitesimal methods evidently corresponded with the atomistic view of natural objects, according to which the great variety of observable phenomena, the endlessly complicated properties of natural objects, could be reduced to a small number of conceivable properties and relations of their smallest parts, and could then be made intelligible and calculable.

The general reader who is unacquainted with the numberless problems and intricate operations of higher mathematics can scarcely realise how in these few words lie really hidden the great questions of all the modern sciences of number and measurement; the trained mathematical student will recognise in a process of inversion not only the rationale of such extensive doctrines as the integral calculus, the calculus of variations, the doctrine of series, the methods of approximation and interpolation, but also the application of analysis to geometry, the theory of curves of higher order, the solution of equations, &c. All these various branches were diligently cultivated by the great mathematicians of the eighteenth century, mostly, however, with the object of solving definite problems which were suggested by the applied sciences,[1]

[1] In general it can be stated that the impetus given to mathematical research by the problems set by the applied sciences has been immeasurably greater than that which can be traced to the abstract treatment of any purely mathematical subject. We have a good example of this at the beginning of the nineteenth century in the great work of Laplace as summed up, for the most part, in the 'Mécanique Celeste' and the 'Théorie des Probabilités,' which contain the beginnings and the development of a great number of purely mathematical theories suggested by problems in astronomy, physics, and statistics. On the other side we have at the same time the so-called "Combinational School" in Germany, whose members and

notably astronomy—not infrequently also as objects of mere curiosity without any practical purpose whatever. In the latter part of the eighteenth century the need was felt of putting the new science into a comprehensive system. The attempts to do this—notably the great text-books of Leonhard Euler in Germany and of Lacroix in France—revealed how uncertain were the foundations and how paradoxical some of the apparent conclusions of the reasoning which, in the hands of the great inventors and masters, had led to such remarkable results.

As in other cases which we dealt with in former chapters of this work, so also in the present instance we may find a guide through the labyrinth of modern mathematical thought in the terms of language around which cluster the more recent doctrines. Two terms present themselves which were rare or altogether absent in older treatises: these terms are the "complex quantity" and the "continuous." To these we can add a third term which we meet with on every page of the writings of mathematicians since Newton and Leibniz, but which has only very recently been subjected to careful analysis and rigorous definition,—the term "infinite." Accordingly we may say that the range of mathematical thought during

11. Modern terms indicative of modern thought.

their labours are almost forgotten, although in their elaborate treatises there are to be found many formulæ which had to be rediscovered when, fifty years later, the general theory of forms and substitutions began to be systematically developed, and proved to be an indispensable instrument in dealing with many advanced mathematical problems. See on the latter subject an article by Major MacMahon on "Combinational Analysis" ('Proc., London Math. Soc.,' vol. xxviii. p. 5, &c.), as also the chapters on this subject and on "Determinants" in the first vol. of the 'Encyclopädie der Mathematischen Wissenschaften' (Leipzig, 1898). Also, *inter alia*, a note by J. Muir in 'Nature,' vol. lxvii., 1903, p. 512.

the last hundred years has grown in proportion to the methodical study and stricter definition of the notions of the complex quantity, of the continuous, and of the infinite. And these conceptions indicate three important logical developments which characterise modern mathematical reasoning. 12. Complex quantities. The conception of the complex quantity or the complex unit introduces us to the possible extension of our system of counting and measuring, retaining or modifying, the fundamental rules on which it is based. 13. The continuous. The conception of the continuous and its opposite, the discontinuous, introduces us to the difference of numbers and quantity, numbers forming a discontinuous series, whilst we conceive all natural changes to be made up of gradual—*i.e.*, of imperceptibly small—changes, called by Newton fluxions. The discussion, therefore, of the continuous leads us ultimately to the question how our system of counting can be made useful for dealing with continuously variable quantities—the processes of nature. 14. The infinite. The conception of the infinite underlies not only the infinitesimal methods properly so called, but also all the methods of approximation by which—in the absence of rigorous measures—mathematical, notably astronomical, calculations are carried out.

Problems involving one or more of these conceptions presented themselves in large number to the analysts of the eighteenth century: there were notably two great doctrines in which they continually occur—the general solution of equations,[1] and the theory of

[1] As it may not be immediately evident how the ideas of continuity have to do with the general solution of equations, I refer to the first publication by Gauss, in 1799, containing a proof of the fundamental theorem of algebra, and its republication fifty years later (see Gauss,

infinite series. The solution of an equation being called finding its roots, it was for a long time assumed that every equation has as many roots as are indicated by its degree. A proof of this fundamental theorem of algebra was repeatedly attempted, but was only completed by Gauss in three remarkable memoirs, which prove to us how much importance he attached to rigorous proofs and to solid groundwork of science. The second great doctrine in which the conceptions of the continuous and the infinite presented themselves was the expansion of mathematical expressions into series. In arithmetic, decimal fractions[1] taken to any number of terms were quite familiar; the infinite series presented itself as a generalisation of this device. A very general formula

15. Doctrine of series. Gauss.

'Werke,' vol. iii. pp. 1 and 71). A very good summary of this proof is given by Hankel ('Complexe Zahlen-Systeme,' p. 87). A purely algebraical demonstration of the same theorem, not involving considerations of continuity and approximations, was also given by Gauss in the year 1816, and reproduced by others, including George Peacock, in his 'Report,' quoted above, p. 297. Hankel (*loc. cit.*, p. 97) shows to what extent Gauss's proof supplemented the similar proofs given by others before and after.

[1] Decimal fractions seem to have been introduced in the sixteenth century. Series of other numbers, formed not according to the decimal but to the dyadic, duodecimal, or other systems, were known to the ancients, and continued in use to the middle ages. The dyadic system was much favoured by Leibniz. It was also known that every rational fraction could be developed into a periodical decimal fraction. Prominent in the recommendation of the use of decimal fractions was the celebrated Simon Stevin, who, in a tract entitled 'La Disme' attached to his 'Arithmétique' (1590, translated into English, 1608), described the decimal system as "enseignant facilement expédier par nombres entiers sans rompus tous comptes se rencontrans aux affaires des hommes." Prof. Cantor ('Gesch. der Math.,' vol. ii. p. 616) says, "We know to-day that this prediction could really be ventured on—that indeed decimal fractions perform what Stevin promised." At the end of his tract he doubts the speedy adoption of this device, connecting with it the suggestion of the universal adoption of the decimal system. The best account of the gradual introduction of decimal fractions is still to be found in George Peacock's 'History of Arithmetic' ('Ency. Metrop.,' vol. i. p. 439, &c.)

of this kind was given by Brook Taylor, and somewhat modified by Maclaurin. It embraced all then known and many new series, and was employed without hesitation by Euler and other great analysts. In the beginning of the century, Poisson, Gauss, and Abel drew attention to the necessity of investigating systematically what is termed the convergency[1] of a series. As a specimen of this kind of research, Gauss published, in 1812, an investigation of a series of very great generality and importance.[2] We can say that through these two isolated memoirs of Gauss, the first of the three on equations, published in 1799, and the memoir on the series of 1812, a new and more rigorous treatment of the infinite and the continuous as mathematical conceptions was introduced into analysis, and that in both he showed the necessity of extending the system of numbering and measuring by the conception of the complex quantity. But it cannot be maintained that Gauss succeeded in impressing the new line of thought upon the science of

[1] A very good account of the gradual evolution of the idea of the convergency of a series will be found in Dr R. Reiff's 'Geschichte der unendlichen Reihen' (Tübingen, 1899, p. 118, &c.) Also in the preface to Joseph Bertrand's 'Traité de Calcul Différentiel' (Paris, 1864, p. xxix, &c.) According to the latter Leibniz seems to have been the first to demand definite rules for the convergency of Infinite Series, for he wrote to Hermann in 1705 as follows: "Je ne demande pas que l'on trouve la valeur d'une série quelconque sous forme finie; un tel problème surpasserait les forces des géomètres. Je voudrais seulement que l'on trouvât moyen de décider si la valeur exprimée par une série est possible, c'est-à-dire convergente, et cela sans connaître l'origine de la série. Il est nécessaire, pour qu'une série indéfinie représente une quantité finie, que l'on puisse démontrer sa convergence, et que l'on s'assure qu'en la prolongeant suffisamment l'erreur devient aussi petite que l'on veut." In spite of this, Leibniz, through his treatment of the series of Grandi, $1-1+1-1$, &c., the sum of which he declared to be $\frac{1}{2}$, seems to have exerted a baneful influence on his successors, including Euler (see Reiff, *loc. cit.*, pp. 118, 158).

[2] The memoir on the Hypergeometrical series.

mathematics in general. This was done about fifteen or twenty years after Gauss had begun to publish his isolated memoirs, in a comprehensive treatment of the subject by Cauchy, who, before 1820, delivered lectures on Analysis at the École Polytechnique and in other colleges, and commenced their publication in 1821. In this course of lectures the discussion of the notions of the infinite, of the continuous, of the convergence of series, and of the extension of our conception of quantity beyond the ordinary or real quantities of algebra, is put in the foreground, and the illicit habit of using the generalisations of algebra without defining the conditions of their validity severely criticised.[1] It is also evident, from the extensive notes which Cauchy added to the "cours" of 1821, that he felt the necessity of a revision of the fundamental notions of algebra. The publication of 1821 was followed by others on the Calculus, and it is through these treatises mainly that a new spirit was infused into general mathematical literature, first in

16. Cauchy's Analysis.

[1] The earliest labours of Cauchy were geometrical, and he evidently acquired through them an insight into the contrast between the rigour of the older geometrical and the looseness of the modern algebraical methods. In this regard he says: "J'ai cherché à leur donner toute la rigueur qu'on exige en géometrie, de manière à ne jamais recourir aux raisons tirées de la généralité de l'algèbre. Les raisons de cette espèce, quoique assez communément admises, surtout dans le passage des séries convergentes aux séries divergentes, et des quantités réelles aux expressions imaginaires ne peuvent être considérés, ce me semble, que commes des inductions propres à faire pressentir quelque fois la vérité, mais qui s'accordent peu avec l'exactitude si vantée des sciences mathématiques. On doit même observer qu'elles tendent à faire attribuer aux formules algébriques une étendue indéfinie, tandis que, dans la réalité, la plupart de ces formules subsistent uniquement sous certaines conditions, et pour certaines valeurs des quantités qu'elles renferment. En déterminant ces conditions et ces valeurs, et en fixant d'une manière précise le sens des notations dont je me sers, je fais disparaître toute incertitude" ('Cours d'Analyse,' 1821, Introd., p. ii).

France, somewhat later also in England and Germany. In the latter country, the highly original writings of Abel, and the independent labours of Jacobi, opened out an entirely new branch of higher mathematics, beginning with the discovery of the property of double periodicity of certain functions.[1] This extensive and fruitful province of analysis for a time retarded the revision and extension of the groundwork of mathematical reasoning which Cauchy had begun, and upon which Gauss evidently desired to make the extension of higher mathematics proceed.[2]

[1] Before the discovery of the functions with a double period, functions with one period were known: the circular and exponential functions—the former possessing a real, the latter an imaginary, period. The elliptic functions turned out to "share simultaneously the properties of the circular functions and exponential functions, and whilst the former were periodical only for real, the latter only for imaginary, values of the argument, the elliptic functions possessed both kinds of periodicity." This great step became clear when it occurred to Abel and Jacobi independently to form functions by inversion of Legendre's elliptic integral of the first kind. The two fundamental principles involved in this new departure were thus the process of inversion and the use of the imaginary, as a necessary complement to the real, scale of numbers. The share which belongs independently to Abel and Jacobi has been clearly determined since the publication of the correspondence of Jacobi with Legendre during the years 1827-32 (reprinted in Jacobi's 'Gesammelte Werke,' ed. Borchardt, vol. i., Berlin, 1881), and of the complete documents referring to Abel, which are now accessible in the memorial volume published in 1902. A very lucid account is contained in a pamphlet by Prof. Königsberger, entitled 'Zur Geschichte der Theorie der Elliptischen Transcendenten in den Jahren 1826-29' (Leipzig, 1879).

[2] Of the four great mathematicians who for sixty years did the principal work in connection with elliptic functions—viz., Legendre (1752-1833), Gauss (1777-1855), Abel (1802-29), and Jacobi (1804-51), each occupied an independent position with regard to the subject,—suggested originally by Euler, and important for the practical applications which it promised. Legendre during forty years, from 1786 onward, worked almost alone: he brought the theory of elliptic integrals, which had occurred originally in connection with the computation of an arc of the ellipse, into a system, and to a point beyond which the then existing methods seemed to promise no further advance. This advance was, however, secured by the labours of Jacobi through the introduction of the novel principles referred to in the last note. Two years before Jacobi's publication commenced, Abel had already approached the subject from an entirely different and much more

That such a revision had become necessary was seen, slowly if in many quarters, but it did not become generally recognised till late in the century, when thinkers of 17. **Revision of fundamentals.**

general point of view. "Abel," as Monsieur L. Sylow says ('Mémorial des études d'Abel,' p. 14), "était avant tout algébriste. Il a dit lui-même que la théorie des équations était son sujet favori, ce qui d'ailleurs apparaît clairement dans ses œuvres. Dans ses travaux sur les fonctions elliptiques, le traitement des diverses équations algébriques dont cette théorie abonde est mis fortement en évidence, et dans le premier de ces travaux, la résolution de ces équations est même indiquée comme étant le sujet principal. Qui plus est, la théorie des équations était entre ses mains l'instrument le plus éfficace. Ce fut ainsi sans aucun doute la résolution de l'équation de division des fonctions elliptiques qui tout d'abord le conduisit à la théorie de la transformation. Elle joue encore un rôle capitale dans sa démonstration du théorème dit théorème d'Abel, et dans les recherches générales sur les intégrales des différentielles algébriques qui se trouvent dans son dernier mémoire le 'Précis d'une Théorie des fonctions elliptiques.'" But whilst Abel certainly took a much more general view than either Legendre or Jacobi, both of whom came to a kind of deadlock on the roads they had chosen (Jacobi, when he attempted to extend the theory of the periodicity of functions), it is now quite clear that Gauss viewed the whole subject almost thirty years before Abel and Jacobi entered the field from a still more general point of view. Already, in 1798, when he was only twenty-one, he must have recognised the necessity of enlarging and defining the fundamental conceptions of algebra and of functionality or mathematical dependence; and it is very likely that the magnitude of the undertaking, for which his astronomical labours left him no time, debarred him from publishing the important results which he had already attained, and which covered to a great extent the field cultivated in the meantime by Abel and Jacobi, leaving only the celebrated theorem of the former (referring to the algebraical comparison of the higher non-algebraical functions) and the discovery of a new function on the part of Jacobi (his Theta function) as the two great additions which we owe to them in this line of research (see Königsberger, *loc. cit.*, p. 104). In this recognition of the fundamental change which mathematical science demanded, and its bearing upon these special problems here referred to, Gauss must have for a long time stood alone; for his great rival Cauchy, to whom we are mainly indebted for taking the first steps in this direction, did not for many years apply his fundamental and novel ideas to the theory of elliptic functions, which up to the year 1844, when Hermite entered the field, were almost exclusively cultivated by German and Scandinavian writers (see R. L. Ellis, "Report on the recent Progress of Analysis," Brit. Assoc., 1846; reprinted in 'Mathematical and other Writings,' p. 311). Nor could it otherwise be explained how Cauchy could keep the manuscript of Abel's great memoir without ever occupying himself with it, and thus delay its publication for fifteen years after it had been presented to the Academy. (See the above-mentioned correspondence between Legendre and Jacobi, 1829; also Sylow, p. 31.)

the highest rank, who for some time had lived apart in the secluded regions of sublime analysis, descended again into the region of elementary science, both pure and applied, where they speedily remodelled the entire mode of teaching. England possessed very early a writer of great eminence who represented this tendency, and whose merits were only partially recognised in his day—Augustus de Morgan.

18. Extension of conception of number.

It will now be necessary to explain more definitely what is meant by the extension of our conception of number and quantity through the introduction of complex numbers or complex quantities. This extension first forced itself on analysts in the theory of equations, then in the algebraical treatment of trigonometrical quantities—*i.e.*, in the measurement of angles, or, as it is now called, of direction in geometry. The first extension of the conception of number lay in the introduction of negative numbers. These admitted of comparatively easy representation arithmetically by counting backward as well as forward from a given datum; practically in the conception of negative possessions, such as debts, geometrically by the two opposite directions of any line in space. In algebra, where the simple operations on quantities are usually preserved in the result and not lost in the simple numerical value of the result as in arithmetic, compound quantities were looked upon as generated by the processes of addition, resulting in the binomial (of which the polynomial was an easy extension), and further by the multiplication with each other of different binomials or polynomials, through which process expressions of higher order or

degree were arrived at. The forward or direct process was easy enough, though even here assumptions or arbitrary rules were included which escaped notice for a long time; but the real labour of the analysts only began with the inverse problem—viz., given any compound quantity, similar in structure to those directly produced by multiplication of binomials, to find the factors or binomials out of which it can be compounded. Now it was found that as in the arithmetical process of division, the invention of fractional quantities; as in that of extraction of roots, the irrational quantities had to be introduced: so in the analysis of compound algebraical expressions into binomial factors, a new quantity or algebraical conception presented itself. It was easily seen that this analysis could be carried out in every case only by the introduction of a new unit, algebraically expressed by the square root of the negative unity. There was no difficulty in algebraically indicating the new quantity as we indicate fractions and irrational quantities; the difficulty lay in its interpretation as a number. Since the time of Descartes geometrical representations of algebraical formulæ had become the custom, and it was therefore natural when once the new, or so-called imaginary, unit was formally admitted, that a geometrical meaning should be attached to it.

Out of the scattered beginnings of these researches two definite problems gradually crystallised: the one, a purely formal or mechanical one—viz., the geometrical representation of the extended conception of quantity, of the complex quantity; the other, a logical 19. The geometrical and the logical problems.

or philosophical one—viz., the clearer definition of the assumptions or principles which underlie arithmetical and algebraical reasoning. And if algebraical, then also geometrical reasoning. Both problems seem to have presented themselves to the youthful mind of Gauss, as is evident from his correspondence with Bessel [1] and Schumacher, and from his direct influence on Bolyai,[2] Möbius, and Von Staudt, perhaps also indirectly on Lobatchevsky.[3] It does not, however, appear as if he

[1] See especially the letters of Gauss to Bessel, dated November and December 1811 and May 1812 ('Briefwechsel,' Leipzig, 1880, p. 151 *sqq.*)

[2] Bolyai, the elder (1775-1856), was a student friend of Gauss in the years 1797 to 1799, and kept up a correspondence with him during half a century. This correspondence has now been published by F. Schmidt and P. Stäckel, Leipzig, 1899, with a supplement containing some information about this extraordinary man. His son, Johann Bolyai (1802-60), is the author of the celebrated "Appendix, scientiam spatii absolute veram exhibens," which was attached to his father's 'Tentamen, juventutem . . . in elementa matheseos puræ . . . introducendi,' 1832. The tract seems to have been written in 1823. A translation, with introduction, has been published by Dr G. Bruce Halsted ('Neomonic Series,' vol. iii. 4th ed., Austin, Texas, 1896). When the elder Bolyai sent to Gauss in the year 1831 to 1832 a copy of his son's tract and of his own work on Geometry, Gauss expressed great surprise at the contents of the former. (See his letter of March 6, 1832.) His remarks that the younger Bolyai had anticipated some of his own ideas on the subject, remind one of a similar remark which he made, May 30, 1828, to Schumacher with reference to Abel's "Memoir on Elliptic Functions" in vol. ii. of Crelle's 'Journal' (see Gauss, 'Werke,' vol. iii. p. 495). In both cases he felt himself relieved from the necessity of publishing his own results, though, so far as those referring to the foundations of geometry are concerned, it does not appear that his ideas had arrived at that state of maturity which the publication of his posthumous papers has proved to have been attained in his treatment of the higher functions. Indeed little or nothing of prime importance has been found among his papers referring to the principles of geometry; and he stated to Bolyai that though he had intended to commit his views to paper, so that they should not be lost, he had not intended to publish anything during his lifetime.

[3] It is doubtful whether Gauss's speculations had any influence on the younger Bolyai's theory, and still more so as regards Lobatchevsky, whose first tract appeared in the 'Kazan Messenger,' 1829 to 1830, but dates back probably to 1826. Inasmuch, however, as the younger Bolyai must have become acquainted

had arrived at any finality in his speculations, and, beyond occasional hints which have only subsequently become intelligible, the love of finish exhibited in all his published writings prevented him from giving to the world the suggestive ideas which evidently formed the groundwork of his mathematical labours. There is no doubt that—like Goethe in a very different sphere—Gauss anticipated individually the developments in the sphere of mathematical thought down to the end of the century. The interpretation of the complex quantity had been given by Wessel, Buée, and Argand [1] in the early years of the century; but it remained unnoticed till it received the sanction of Gauss in a celebrated memoir referring to the theory of numbers, and until in

through his father with the speculations of the youthful Gauss, and as Lobatchevsky was a pupil of another student friend of Gauss in the person of Prof. Bartels, it is not unlikely that the interest which these thinkers took in the subject can be originally traced to the same source. (See Dr Halsted's address on Lobatchevsky, 'Neomonic Series,' vol. i., 1894.) A complete bibliography of the earlier papers, referring to the so-called "non-Euclidean" literature down to 1878, is given by Dr Halsted in the first two vols. of the 'American Journal of Mathematics': the most recent publications are those of the Hon. B. A. W. Russell in his work, 'The Foundations of Geometry' (1897) and his excellent article on "Non-Euclidean Geometry" in the 28th vol. of the 'Ency. Brit.' See also Klein's lithographed lectures on 'Nicht-Euklidische Geometrie,' Göttingen, 1893.

[1] The first somewhat exhaustive historical statement as to the geometrical representation of the complex or imaginary quantity was given by Hankel in the above-mentioned work (see above, note, p. 645), p. 82. He there says, after discussing the claims of others,—notably of Gauss,—that Argand in his 'Essai' of the year 1806 (re-edited by Hoüel, 1874) "had so fully treated of the whole theory that later nothing essentially new was added, and that, except a publication of still earlier date were found, Argand must be considered the true founder of the representation of complex quantities in the plane." Such an earlier publication has indeed been met with in a tract by Caspar Wessel, which was presented to the Danish Academy in 1797, and published in 1799. Having been overlooked, like Argand's 'Essai,' it has now been republished at Copenhagen, 1897, with the title 'Essai sur la représentation de la direction' (see 'Encyk. Math. Wissenschaften,' vol. i. p. 155).

this country the labours of De Morgan and of Sir William Rowan Hamilton gave the matter a further and very important extension.[1] It was also in this country that the second problem, the critical examination of the principles which underlie the process of legitimate generalisation of algebra, received distinct attention. To George Peacock, and to the school of algebraists which followed him, is due the merit of having brought out clearly the three fundamental laws of symbolical reasoning now generally admitted in text-books on the subject—the associative, distributive, and commutative principles. That these principles were to a great extent conventional, or empirically adopted from ordinary arithmetic, and in consequence not necessarily indispensable for a consistent system of symbolical reasoning, has been generally admitted ever since Sir William Rowan Hamilton, after ten years of labour, succeeded in establishing a new calculus—the method of quaternions, in which the commutative principle of multiplication is dropped. This

20. Quaternions.

[1] Far more important than the suggestions or artifices mentioned in the foregoing note, and which since the time of Argand and Gauss have been variously modified, is the conception that our common numbers do not form a complete system without the addition of the imaginary unit, but that with the introduction of a second unit "numbers form a universe complete in itself, such that, starting in it, we are never led out of it. There may very well be, and perhaps are, numbers in a more general sense of the term; but in order to have to do with such numbers (if any) we must start with them" (Cayley in art. "Equation," 'Ency. Brit.'; 'Coll. Works,' vol. xi. p. 503). There seems little doubt that this conception was first clearly established in the mind of Gauss, and that none of the contemporary writers can be shown to have had a similarly clear insight. Since this has become generally recognised—and we owe this recognition probably to the independent labours of Grassmann and Riemann—the discussion of the whole subject has been raised to a much higher level, as may be seen by comparing the Report of Peacock, quoted above, with the discussion of Hankel (*loc. cit.*), and still more with the exhaustive article by Prof. E. Study in vol. i., 'Encyk. Math. Wiss.,' pp. 147-184.

calculus was shown to be of special use in expressing the relations of spherical trigonometry. Two terms expressing definite notions special to geometry, by which science has been enriched and practical application greatly simplified, are an outcome of this line of research. These are the terms "vector," to express the notion of directed magnitude—*i.e.*, of direction and magnitude combined as distinguished from magnitude and position alone; and the notion of an "operator" which changes direction and magnitude as an ordinary multiplier changes magnitude only.[1] It was shown by Argand and others that the

[1] These two notions, which have their origin in the writings of Hamilton on the one side and the Calculus of Operations on the other, belong to this country and to a period during which mathematical researches were carried on in a fragmentary manner, and much out of contact with the contemporary mathematics of the Continent. Both the Calculus of Quaternions of Hamilton and the Calculus of Operations were looked upon for a long time as curiosities (as was also the Barycentric Calculus of Möbius in Germany). Gradually, however, the valuable ideas which were contained in them became recognised as much from the practical as from the theoretical point of view. In the former interest the application of Vector Analysis or the Algebra of Directed Quantities received a great impetus when the need was felt of having an algebra of "physical quantities." This found expression in the writings of Clerk-Maxwell. (See his 'Treatise on Electricity and Magnetism,' vol. i. p. 8, 2nd ed., as also his paper on "The Mathematical Classification of Physical Quantities," 1871. 'Coll. Papers,' vol. ii. p. 257.) In the practical application of electrical theories these notions have since become indispensable, and the subject has received increasing attention, notably in America, which holds a foremost place in the development of electrical science and its application. Mathematicians of the first order, such as J. Willard Gibbs, have published text-books on the subject, whilst other electricians of eminence, such as Mr Oliver Heaviside, have elaborated special forms of the Directional Calculus to serve their purposes. In Dynamics the Dublin School, represented after the death of Hamilton by Sir Robert S. Ball (in his 'Theory of Screws,' 1876), has had an important influence in the introduction of novel and more appropriate methods which have gradually permeated the general treatment of the subject. Whilst there is no doubt that for a long time the Calculus of Quaternions was the only methodical elaboration of these novel and useful ideas, it was overlooked that simultaneously and quite independently H. Grassmann of Stettin (see above, vol. i. p. 243) had worked out a much more comprehensive and fundamental calculus, of which the method of quaternions and all the different forms of Vector Analysis can be

arithmetic based upon two units instead of one—*i.e.*, the arithmetic of couples or complex quantities—could be completely and consistently represented by choosing as axes whereon the separate units were counted, the two perpendicular axes of Cartesian geometry. An attempt to extend this geometrical representation into space led Hamilton to the invention of his method, Gauss having very early satisfied himself that within the limits of ordinary algebra no further extension was necessary or possible.

21. Foundations of geometry.

The examination into fundamental principles was not limited in the mind of Gauss to those of algebra: he early applied himself likewise to those of geometry and of dynamics. The great French mathematicians, such as Legendre and Lagrange, were also occupied with such speculations. They have been carried on all through the century, but have only towards the end of the period been brought into connection and shown to be of importance for the general progress of mathematics. The secluded, and for a long time unappreciated, labours of isolated but highly original thinkers have accordingly

considered as merely special instances. This has now been abundantly proved through the writings of mathematicians in all countries, among whom I will only mention Hankel and Dr V. Schlegel in Germany, Clifford, Prof. Henrici, and latterly Mr Whitehead in England, Prof. Peano in Italy, and M. Burali Forti in France. See on the whole subject, on the fate of Grassmann and of his great work, V. Schlegel, 'Die Grassmann'sche Ausdehnungslehre,' Leipzig, 1896; also, by the same author, a short biography of Grassmann (Leipzig, Brockhaus, 1878). A complete edition of Grassmann's works is being published by Teubner. Those who are interested in seeing how the notions underlying the directional calculus are gradually becoming clarified, and the terminology and notation settled, may read with profit the controversy carried on in the pages of 'Nature,' vols. xlvii. and xlviii., between Prof. Macfarlane, Willard Gibbs, Mr O. Heaviside, Mr A. M'Aulay, and Dr Knott; also Dr Larmor's review of Hayward's 'Algebra of Coplanar Vectors' (vol. xlvii. p. 266), and Sir R. S. Ball's reference to the 'Ausdehnungslehre' of Grassmann (vol. xlviii. p. 391, 1893).

received tardy recognition. Such speculations can be carried on either as fascinating exercises of mere ingenuity, or for practical purposes to improve the refined instruments of mathematical calculation, or in the philosophical interest of arriving at the fundamental processes of human thought and intuition.[1] Many persons think that only the second of these three in-

[1] Already Euler had remarked on the different interests that prompted mathematical research. Referring to the writings of Count Fagnano, he says in the introduction to the first of his memoirs on Elliptic Integrals (1761, quoted by Brill & Nöther in 'Bericht der Deutschen Mathematiker-Vereinigung,' vol. iii. p. 206): "If one looks at mathematical speculations from the point of view of utility, they can be divided into two classes: first, those which are of advantage to ordinary life and other sciences, and the value of which is accordingly measured by the amount of that advantage. The other class comprises speculations which, without any direct advantage, are nevertheless valuable because they tend to enlarge the boundaries of analysis and to exercise the powers of the mind. Inasmuch as many researches which promise to be of great use have to be given up owing to the inadequacy of analysis, those speculations are of no little value which promise to extend the province of analysis. Such seems to be the nature of observations which are usually made or found *a posteriori*, but which have little or no chance of being discovered *a priori*. Having once been established as correct, methods more easily present themselves which lead up to them, and there is no doubt that through the search for such methods the domain of analysis may be considerably extended." The school of mathematicians headed by Abel and Jacobi pursued mathematics from purely scientific interest, and was criticised on this ground by eminent contemporary mathematicians in France: see a letter of Jacobi to Legendre, dated July 2, 1830, in which he refers to a Report of Poisson on his great work, but adds: "M. Poisson n'aurait pas dû reproduire dans son rapport une phrase peu adroite de feu M. Fourier où ce dernier nous fait des reproches, à Abel et à moi, de ne pas nous être occupés de préférence du mouvement de la chaleur. Il est vrai que M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels ; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain et que sous ce titre, une question de nombres vaut autant qu'une question du système du monde." In the sequel he adds : "Je crois entrevoir que toutes ces transcendantes" (*i.e.*, the elliptic and Abelian functions) "jouissent des propriétés admirables et inattendues auxquelles on peut être conduit par le théorème d'Abel. . . . J'ai réfléchi aussi de temps en temps sur une méthode nouvelle de traiter les perturbations célestes, méthode dans laquelle doivent entrer les théories nouvelles des fonctions elliptiques."

ducements is likely to prove fruitful for the progress of science; they look upon the first as an amusing pastime, and upon the third as empty and not devoid of danger. In recognition of the partial correctness of this view, I will follow up the practical stimulus in its fruitful influence upon the development of the lines of mathematical research.

This stimulus came in the closing years of the preceding century through the lectures of Gaspard Monge at the École Normale, and has become popularly known
22. Descriptive Geometry. through his invention of Descriptive Geometry, the first modern systematic application of purely graphical methods in the solution of mathematical problems. As Cauchy was the founder of the modern school of analysts, so Monge, together with Carnot, founded the modern school of geometricians; Dupin, Poncelet, and Chasles being among his most illustrious pupils. The aim of this school was to give to geometrical methods, such as had been practised by the ancients,[1] the same generality and systematic unity which characterised the analytical methods introduced by Descartes.

Not long after the introduction of the latter, Leibniz

[1] These methods had been largely used in this country by Newton, Robert Simson, and Stewart. They were systematised by L. N. M. Carnot. Chasles ("Discours d'inauguration, &c.," 1846, 'Géométrie Supérieure,' p. lxxvii) says: "Dans le siècle dernier, R. Simson et Stewart donnaient, à l'instar des Anciens, autant de démonstrations d'une proposition, que la figure à laquelle elle se rapportait présentait de formes différentes, à raison des positions relatives de ses diverses parties. Carnot s'attacha à prouver qu'une seule démonstration appliquée à un état assez général de la figure devait suffire pour tous les autres cas; et il montre comment, par des changements de signes de termes, dans les formules démontrées par une figure, ces formules s'appliquaient à une autre figure ne différant de la première, commes nous l'avons dit, que par les positions relatives de certaines parties. C'est ce qu'il appela le 'Principe de corrélation des figures.'"

had foretold[1] the possibility and necessity of such an independent development of pure geometry, in which the relations of position in space, as opposed to those of measure, magnitude, or quantity, would be placed in the foreground. Projection, as practised in the drawing of maps, and perspective, as practised in the fine and descriptive arts, had already revealed a number of remarkable properties of figures in the plane and in space. By continuous motion of points or lines, by artifices like throwing of shadows, by sections of solids with lines and surfaces, a vast number of problems had been solved and isolated theorems established. The method here practised was that of construction, as in analysis the method was that of calculation with subsequent interpretation. All this purely constructive work was to be brought together and systematically combined in a whole. It was evidently a distinct line of research, based upon intellectual processes other than the purely analytical method—a line which, as it seemed to its followers, had been unduly neglected and pushed into the background. Although Monge became the founder of this purely descriptive or constructive branch of geometry, he was himself equally great as an analyst; in fact, the fusion in his mind of the two methods was the origin of much of his greatest work. In attempting to carry out more thoroughly the separation or independent development of the constructive or descriptive method, his great pupil,
J. V. Poncelet—whilst deprived of all literary resources 23. Poncelet.

[1] See the quotations from his letters to Huygens and others given above, vol. i. p. 103 note.

in the prisons of Russia—meditated on the real cause of the power which algebraical analysis possessed, on the reason why geometry proper was deprived of it, and what might be done to give it a similar generality. In pursuing this line of thought he was led to discover the cause of the existing limitation of purely geometrical reasoning in its rigidity, inasmuch as it was arrested as soon as its objects ceased to have a positive or absolute, that is a physical, existence.[1] Opposed to this limitation was the freedom of the analytical method, which, operating with indeterminate symbols, could, by letting them change gradually, include not only what was explicitly given, but also that which was merely implied; not only the finite, but likewise the infinite; not only the real, but likewise the fictitious or imaginary. In order to gain a similar generality in purely geometrical or descriptive science, a similar flexibility would have to be introduced. Poncelet was thus led to the enunciation of his celebrated and much-criticised "principle or law of continuity."[2]

[1] See the "Introduction" to the 1st volume of the 'Traité des Propriétés projectives des figures,' pp. xi, xii. I quote from the 2nd edition of 1865. The 1st was published in 1822. The researches date from 1813, the year of Poncelet's imprisonment. See "Préface de la première édition."

[2] Ibid., Introduction, p. xiv. On the principle of continuity in geometry, see an article in vol. xxviii. 'Ency. Brit.' by the Rev. Charles Taylor, and the references given therein; also Prof. E. Kötter's Report on the "Development of Synthetic Geometry" in vol. v. of the 'Jahresbericht der Deutschen Mathematiker Vereinigung,' p. 122, &c.: "Originally the expositions referring to the principle of continuity were intended to occupy much greater space. . . . In consequence of correspondence with Terquem, Servois, and Brianchon, Poncelet desisted from the publication of it. . . . However cautiously Poncelet advanced his principle"—in the 'Essai sur les propriétés projectives des sections coniques' (presented to the Academy in 1820)—"it nevertheless aroused the doubts of Cauchy, who in his report on Poncelet's paper warns against the too hasty application of the principle. Gergonne accompanied the reprint of this report with notes, in which he characterised.

Analytical geometry, by substituting an algebraical expression for a geometrical figure—say a curve,—could apply to it all the artifices of abstract analysis. By varying the co-ordinates you can proceed along the whole extent of the curve and examine its behaviour as it vanishes into infinity, or discover its singular points at which there occurs a break of continuity: you can vary its constants or parameters, and gradually proceed from one curve to another belonging to the same family, as is done in grouping together all curves of the second order, or—as was done in the calculus of variation, invented by Euler and Lagrange—you can vary the form of the equation, proceeding from one class of curve to another. Now clearly all this operating on equations and symbolic expressions was originally abstracted from geometry, including the mechanical conception of motion; in particular the ideas which underlie the method of fluxions were suggested by the motion of a point in space. The conception of continuous motion in space—

the principle as a valuable instrument for the discovery of new truths, which nevertheless did not make stringent proofs superfluous." Cauchy's report seems to have aroused Poncelet's indignation. Hankel ('Elemente der Projectivischen Geometrie,' 1875, p. 9) says: "This principle, which was termed by Poncelet the 'Principle of Continuity,' inasmuch as it brings the various concrete cases into connection, could not be geometrically proved, because the imaginary could not be represented. It was rather a present which pure geometry received from analysis, where imaginary quantities behave in all calculations like real ones. Only the habit of considering real and imaginary quantities as equally legitimate led to that principle which, without analytical geometry, could never have been discovered. Thus pure geometry was compensated for the fact that analysis had for a long time absorbed the exclusive interest of mathematicians; indeed it was perhaps an advantage that geometry, for a time, had to lie fallow." Kötter continues: "Von Staudt was the first who succeeded in subjecting the imaginary elements to the fundamental theorem of projective geometry, thus returning to analytical geometry the present which, in the hands of geometricians, had led to the most beautiful results."

of motion of points, lines, planes—corresponded accordingly to the notion of variability in analysis. The introduction of motion, gradual and continuous, would give to purely geometrical or descriptive reasoning the same flexibility which analysis had acquired in the calculus of fluxions and of variations. Figures would lose their rigidity and isolation and limited nature and become movable, related to each other, filling the whole of space instead of a restricted and confined area or region. It is the peculiarity of the modern as opposed to the older geometry, never to let figures become motionless or rigid,[1] never to consider them in their isolation, but always in their mutual relations; never to have regard only to a finite portion of a line or surface, but to conceive of it in its infinite extension. By a reaction of analysis and geometry on each other, freedom and generality have been gradually acquired.

24. Character of modern geometry.

But this moving about of figures in space in order to learn their properties and mutual relations must be according to some method; otherwise it will not lead to scientific and exact knowledge. Poncelet, in considering how the two successful methods in geometry — the Cartesian and the Descriptive — had attained to their perfection, discovers a general principle which underlies their proceedings, and which is capable of great extension: this is the principle of projection.[2]

[1] See, *inter alia*, what Geiser says of Jacob Steiner's method in his pamphlet 'Zur Erinnerung an Jacob Steiner,' Schaffhausen, 1874, p. 27.

[2] 'Traité des Propriétés projectives,' vol. i. p. xviii: "En réfléchissant attentivement à ce qui fait le principal avantage de la Géométrie descriptive et de la méthode des coordonnées, à ce qui fait que ces branches des Mathématiques offrent le caractère d'une véritable doctrine, dont les principes, peu nombreux, sont liés et enchaînés d'une manière nécessaire

Of this principle of projection, which Poncelet at once introduces in the more general form as conical or central projection, two signal applications existed in the treatises on Conic Sections handed down from antiquity, and in the practical methods and Rules of Perspective invented by Lionardo da Vinci and further developed by various geometricians. The results, which lay scattered in many books and memoirs, Poncelet collected in a systematic form, bringing them, by the application of the law of continuity, under a few general and eminently useful points of view or principles. By the method of projection or perspective he "transformed figures which are very general into others which are particular, and *vice versa*." He established the principle of "homology" in figures, and by showing how figures apparently very different could be described by the process of projection from the same original figure, he showed that there existed a peculiar relation among figures—viz., their "reciprocity." [1] 25. Method of projection.

et par une marche uniforme, on ne tarde pas à reconnaître que cela tient uniquement à l'usage qu'elles font de la projection."

[1] The properties of figures, called by Poncelet "homology" and "reciprocity," refer to the correspondence of certain elements of one figure to those of another figure. In the case of "homology," we have to do with corresponding points or corresponding lines—*i.e.*, with the correspondence of the same elements. In the case of "reciprocity," we have to do with correspondence of points or lines in the one figure, with lines or points in the other—*i.e.*, with the correspondence of different elements. The idea of placing figures in an homologous relation was got by the device of making two planes, which contained figures in perspective, fall together into one plane; upon which the section of the two original planes became the "axis," and the eye-point the "centre" of homology—all situated in one and the same plane. Poncelet had already conceived of the possibility of reducing the two planes in Monge's 'Descriptive Geometry,' which represent the plan and elevation of a figure in one plane, on which the elevations were marked by what are now called "contour lines." The idea of the correspondence of figures by what is called "reciprocity" was sug-

26. Law of continuity. By the law of continuity he showed how in pure geometry it became necessary to introduce the consideration of points and lines which vanish into infinity or which become imaginary, establishing by their invisible elements the continuous transition from one geometric form to another; just as in algebra these conceptions had forced themselves on the attention of analysts. Ideal elements were thus made use of to lead to the discovery of real properties.

The consideration of lines and points which vanish or lie at infinity was familiar to students of perspective from the conception of the "vanishing line"; but the inclusion of ideal points and lines was, as Hankel says, a gift which pure geometry received from analysis, where imaginary (*i.e.*, ideal or complex) quantities behave

27. Ideal elements. in the same way as real ones. Without the inclusion of these ideal or invisible elements the generality or continuity of purely geometrical reasoning was impossible.

The geometrical reasoning of Monge, Carnot, and Poncelet was thus largely admixed with algebraical or analytic elements. It is true that Monge's descriptive geometry was a purely graphical method, and that

gested to Poncelet by the property, known already to De la Hire ("Sectiones Conicæ," 1685), that in the plane of a conic section every point corresponds to a straight line called its "polar," that to every straight line corresponds a point called its "pole," that the "polars" corresponding to all the points of a straight line meet in one and the same point, and *vice versa* that the "poles" corresponding to all lines going through one and the same point lie on a straight line; the line and point in question standing in both cases in the relation of pole and polar to each other. Poncelet uses "this transformation of one figure into its reciprocal polar systematically as a method for finding new theorems: to every theorem of geometry there corresponds in this way another one which is its 'polar,' and the whole of geometry was thus split up into a series of truths which run parallel and frequently overlap each other" (Hankel, *loc. cit.*, p. 20).

Poncelet's method of central projection attacked geometrical problems from a purely constructive point of view. Nevertheless the frequently expressed object of the later writings of Monge, as well as those of Carnot and Poncelet, was to introduce into geometrical reasoning the generality and continuity which analysis possessed, and this was largely attained by the interpretation of notions taken over from analysis. Their endeavours were, however, in the sequel crowned by the discovery of a purely geometrical property, the understanding of which has ever since formed the basis of what may be termed modern geometry.

This remarkable property, which may be regarded as revealing the very essence of extension in space or of the "space-manifold,"—inasmuch as it brings the different elements of space into mutual relation,—is the so-called principle of "duality" or of "reciprocity." The principle of duality is now usually defined to mean that in geometry on the plane or in space, "figures coexist in pairs, two such coexisting figures having the same genesis and only differing from one another in the nature of the generating element."[1] The elements of plane geometry are the point and the line; the elements of solid geometry are the point and the plane. By interchanging these correlative terms, correlative propositions may be written down referring to plane and to space geometry. In projective geometry there are two processes which are correlative or complementary to each other—the process of projection and the process of section. We can project 28. Principle of duality.

[1] Cremona, 'Elements of Projective Geometry,' transl. by Leudesdorf. Oxford, 1885, p. 26.

from a point drawing lines or rays on the plane and in space, and we can cut these by lines in a plane or by planes in space. And it can be shown that "if one geometric form has been derived from another by means of one of these operations, we can conversely, by means of the complementary operation, derive the second from the first."[1]

29. Reciprocity.

The projective geometry of Poncelet contains the twofold origin of the principle of duality in his method of projection and section, and in his theory of the reciprocity of certain points and lines in the doctrine of conic sections, called the theory of reciprocal polars. But the mathematician who first expressed the principle of duality in a general—though not in the most general—form was Gergonne, who also recognised that it was not a mere geometrical device but a general philosophical principle, destined to impart to geometrical reasoning a great simplification. He sees in its enunciation the dawn of a new era in geometry.[2]

[1] Cremona, *loc. cit.*, p. 33.

[2] The principle of Duality seems to have been first put forward in its full generality by Gergonne, inspired probably by the theory of Reciprocal Polars (see note, p. 663) enunciated by Poncelet, who many years afterwards carried on a voluminous polemic as to the priority of the discovery. "Gergonne saw that the parallelism (referred to above) is not an accidental consequence of the property of conic sections, but that it constitutes a fundamental principle which he termed the 'principle of duality.' The geometry which is usually taught, and in which a line is considered to be generated by the motion of a point, is opposed by another geometry equally legitimate in which a point is generated by the rotation of a line. Whereas in the first case the line is the locus of the moving point, in the latter case the point is the geometrical intersection of the rotating line. In this generality the principle of duality has been incorporated into modern geometry" (Hankel, *loc. cit.*, p. 21). Gergonne says of the new principle (1827, see Supplement to vol. ii. 2nd ed. of Poncelet's 'Traité,' p. 390): "Il ne s'agit pas moins que de commencer pour la géométrie, mal connue depuis près de deux mille ans qu'on s'en occupe, une ère tout-à-fait nouvelle; il s'agit d'en mettre tous les anciens traités à peu près au

It must, however, in all fairness be stated that about the period from 1822 to 1830 this great simplification and unification of geometric science was as it were in the air—that it had presented itself to various great thinkers independently, being suggested from different points of view. The beginnings can no doubt be traced in the beautiful theorems of older French mathematicians, such as Pascal and De la Hire, and more generally in the suggestive methods of Monge and Poncelet; its first formal enunciation is in the memoirs of Gergonne: but the comprehensive use of it—the rewriting of geometry from this point of view—was the idea of Jacob Steiner, 30. Steiner.
who, in his great but unfinished work on the "Systematic Development of the Dependence of Geometric Forms" (1830), set himself the great task "of uncovering the organism by which the most different forms in the world of space are connected with each other." "There are," he says, "a small number of very simple fundamental relations in which the scheme reveals itself, by which the whole body of theorems can be logically and easily developed." "Through it we come, as it were, into possession of the elements which Nature employs with the greatest economy and in the simplest manner in order to invest figures with an infinite array of properties."[1]

rebut, de leur substituer des traités d'une forme tout à fait différente, des traités vraiment philosophiques qui nous montrent enfin cette étendue, réceptacle universel de tout ce qui existe, sous sa véritable physionomie, que la mauvaise méthode d'enseignement adoptée jusqu'à ce jour ne nous avait pas permis de remarquer; il s'agit, en un mot, d'opérer dans la science une révolution aussi impérieusement nécessaire qu'elle a été jusqu'ici peu prévue."

[1] See the Preface to the 'Systematische Entwickelung, &c.,' in Jacob Steiner's 'Gesammelte Werke' (ed. Weierstrass), vol. i. p. 229. "In the beautiful theorem that a conic section can be generated by the intersection of two projective pencils (and the dually

The labours of Poncelet and Steiner introduced into geometry a twofold aspect, and accordingly, about the middle of the century, we read a good deal of the two kinds of geometry which for some time seemed to develop independently of each other. The difference has been defined by the terms "analytic or synthetic," "calculative or constructive," "metrical or projective." The one operated with formulæ, the other with figures; the one studied the properties of quantity (size, magnitude), distances, and angles, the other those of position.

The projective method seemed to alter the magnitude of lines and angles and retain only some of those of position and mutual relation, such as contact and intersection. The calculating or algebraical method seemed to isolate figures and hide their properties of mutual interdependence and relation.

31. Mutual influence of metrical and projective geometry.

These apparent defects stimulated the representatives of the two methods to investigate more minutely their hidden causes and to perfect both. The algebraical formula had to be made more pliable, to express more naturally and easily geometrical relations; the geometrical method had to show itself capable of dealing with quantitative problems and of interpreting geometrically those modern notions of the infinite and the complex which the analytic aspect had put promi-

correlated theorem referring to projected ranges), Steiner recognised the fundamental principle out of which the innumerable properties of these remarkable curves follow, as it were, automatically with playful ease. Nothing is wanted but the combination of the simplest theorems and a vivid geometrical imagination capable of looking at the same figure from the most different sides in order to multiply the number of properties of these curves indefinitely" (Hankel, *loc. cit.*, p. 26; see also Cremona, 'Projective Geometry,' p. 119).

nently into the foreground. The latter was done by the geometric genius of Von Staudt, who succeeded in giving a purely geometrical interpretation of the imaginary or invisible elements[1] which algebra had introduced, whilst Steiner astonished the mathematical world by the fertility of the methods by which he solved the so-called isoperimetrical problems—*i.e.*, problems referring to largest or smallest contents contained in a given perimeter or *vice versâ*, problems for which Euler and Lagrange had invented a special calculus.[2] In spite of

[1] The geometrical interpretation of the imaginary elements is given by Von Staudt in a sequel to his 'Geometrie der Lage' (1847), entitled 'Beiträge zur Geometrie der Lage' (1856-60); and after having been looked upon for a long time as a curiosity or a "hair-splitting abstraction," it has latterly, through the labours of Prof. Reye ('Geometrie der Lage,' 1866-68) and Prof. Lüroth ('Math. Annalen,' vol. xiii. p. 145), become more accessible, and is systematically introduced into many excellent text-books published abroad. The simplest exposition I am acquainted with is to be found in the later editions of Dr Fiedler's German edition of Salmon's 'Conic Sections' (6th Aufl., vol. i. p. 23, &c., and p. 176, &c.) In 1875, before the great change which has brought unity and connection into many isolated and fragmentary contributions had been recognised, Hankel wrote with regard to Von Staudt's work, and in comparision with that of Chasles, as follows: "The work of Von Staudt, classical in its originality, is one of those attempts to force the manifoldness of nature with its thousand threads running hither and thither into an abstract scheme and an artificial system: an attempt such as is only possible in our Fatherland, a country of strict scholastic method, and, we may add, of scientific pedantry. The French certainly do as much in the exact sciences as the Germans, but they take the instruments wherever they find them, do not sacrifice intuitive evidence to a love of system nor the facility of method to its purity. In the quiet town of Erlangen, Von Staudt might well develop for himself in seclusion his scientific system, which he would only now and then explain at his desk to one or two pupils. In Paris, in vivid intercourse with colleagues and numerous pupils, the elaboration of the system would have been impossible" (*loc. cit.*, p. 30).

[2] See the lecture delivered by Steiner in the Berlin Academy, December 1, 1836, and the two memoirs on 'Maximum and Minimum' (1841), reprinted in 'Gesammelte Werke,' vol. ii. p. 75 *sqq.*, and 177 *sqq.*, especially the interesting Introductions to both, in which he refers to his forerunner Lhuilier (1782), deploring that others had needlessly forsaken the simple synthetical methods adopted by him. Some of Steiner's expositions in these matters were apparently so easy that non-mathematical listeners

these marvellous works of genius, science is probably indebted for its greatest advances to those mathematicians who, like Plücker in Germany, Chasles in France, and Cayley in England, employed the analytic and constructive methods alternately and with equal mastery.

It is impossible—and it is not my object—to allot to each of these original thinkers the special ideas introduced by him into modern science; but for the purpose

like Johannes Müller could not understand how such simple things could be brought before the Academy of Sciences, whereas the great mathematician Dirichlet was full of praise of the ingenuity of the method by which problems were solved which the Calculus of Variations attacked long after Steiner, and then only in ways which the synthetical method had indicated (see Geiser, 'Zur Erinnerung an Jacob Steiner,' p. 28). It must not be supposed, however, that Steiner was an extreme purist so far as geometrical methods were concerned, for he says himself "that of the two methods neither is entitled to exclude the other; rather both of them will, for a long time, have plenty to do in order to master the subject to some extent, and then only can an opinion as to their respective merits be formed" ('Ges. Werke,' vol. ii. p. 180). An instance of a celebrated problem being treated alternately by synthetic and analytic methods is that of the Attraction of Ellipsoids, in which the Theorem of Maclaurin had created quite a sensation. In spite of the admiration which it evoked, both Legendre and Poisson expressed the opinion that the resources of the synthetic method are easily exhausted. The latter, whilst admitting "que la synthèse ait d'abord devancé l'analyse," nevertheless concludes that "la question n'a été enfin résolue complètement que par des transformations analytiques . . . auxquelles la synthèse n'aurait pu suppléer." This expression of opinion was falsified when Chasles presented to the Academy, in the year 1837, a memoir in which, through the study of confocal surfaces, the Theory of Maclaurin was synthetically proved in its full generality. Poinsot, who reported on this memoir, attached the following remarks: "Ce mémoire remarquable nous offre un nouvel exemple de l'élégance et de la clarté que la géométrie peut répandre sur les questions les plus obscures et les plus difficiles. . . . Il est certain qu'on ne doit négliger ni l'une ni l'autre; elles sont au fond presque toujours unies dans nos ouvrages, et forment ensemble comme l'instrument le plus complet de l'esprit humain. Car notre esprit ne marche guère qu'à l'aide des signes et des images; et quand il cherche à pénétrer pour la première fois dans les questions difficiles, il n'a pas trop de ces deux moyens et de cette force particulière qu'il ne tire souvent que de leur concours. C'est ce que tout le monde peut sentir, et ce qu'on peut reconnaître dans le Mémoire même." (Chasles, 'Rapport sur les progrès de la géometrie,' 1870, p. 105, &c.)

of bringing some order into the tangled web of mathematical speculation, mainly represented by these, I shall identify the name of Plücker with the great advance which has taken place in geometry through the change in our ideas as to the elements of space construction and the generalisation of our ideas of co-ordinates: with Chasles I shall specially connect the modern habit in geometry of combining figures in finite space with their infinitely distant elements, and with Cayley the application to geometrical science of the novel and comprehensive methods of modern algebra. Let us dwell for a moment on each of these three great departures.

32. Plücker, Chasles, Cayley.

The elements of any science are a very different thing from the elements of the special object with which that science is concerned. The elements of chemistry are not the chemical elements. The latter are, we suppose, something existing in nature, something fixed and unalterable, which science aims at finding out; the former are certain conceptions from which we find it convenient to start in teaching, expounding, and building up the science of chemistry. The latter are artificial, the former are natural. The same remark obtains in geometrical science. The elements of geometry have an historical, a practical beginning: the elements of space form a conception which gradually emerges in the progress of geometrical science. In every science there is a tendency to replace the casual and artificial elements by the natural or real elements, and to build up the historical traditional body of doctrine anew, using the very elements which Nature herself, as it were, employs in producing her actual forms and objects. As the pass-

33. Historical and logical foundations.

age quoted above shows, such an idea must have been before the mind of Jacob Steiner when he wrote the 'Systematische Entwickelung.' Through Euclid geometricians had learnt to begin with the straight line of definite—not indefinite—length, the triangle, the circle, advancing to more complicated figures; practice had made geometry a science of mensuration, involving number; the convenience of practice in astronomy, geodesy, and geography had introduced the artifice of referring points and figures in space to certain arbitrarily chosen data—points and lines. The terms "right ascension" and "declination," "altitude" and "azimuth," "latitude" and "longitude," led to the co-ordinates of Descartes and to analytical geometry. In this older and modern geometry, the beginnings were arbitrary, and many conceptions were introduced which were foreign to the object of research. It was through a slow process that in quite recent times—notably during the nineteenth century — mathematicians became aware how artificial were their methods, and with how many foreign elements they had encumbered the objects of their study. To replace the artificial by natural conceptions, and to open the eyes of geometricians to the advantage of not confining themselves to the point (its motion and distances) as the element in their space construction, no one did more than Julius Plücker of Bonn. We have now not only a point-geometry, but likewise a line-geometry—*i.e.*, we have a geometry in which the line is the primary element, the point being the secondary element, defined by the intersection of two lines. This conception, which

can be applied also to geometry in space, the point being conceived as generating a plane by its motion, or three planes defining a point by their intersection, leads us to the same idea of dual correspondence or reciprocity which Poncelet and Gergonne had arrived at by entirely different considerations. Plücker's was an analytical mind, and with him the principle of duality at once assumes an analytical form. He saw that the same equation lent itself to a twofold interpretation, accordingly, as we adopt point co-ordinates or line co-ordinates —*i.e.*, according as we refer our geometrical figure to the point or the line as the moving and generating space element. Through this step the idea of co-ordinates was generalised, and the dualistic conception of figures in space received an analytical expression. It was the junction of analytical and descriptive methods on a higher level, from which an entirely novel and fertile development of geometry became possible.

34. Generalised co-ordinates.

Whilst the labours of Plücker lay in the direction of making analytical formulæ more natural, better adapted to the expression of geometrical forms and relations, and of reading out of these remodelled formulæ novel geometrical properties, the French school, with Michel Chasles [1]

[1] In addition to numerous valuable memoirs, Chasles published, among others, two works of paramount importance, inasmuch as they for a long time dominated purely geometrical research, not only in France but also in Germany and England,—the 'Aperçu historique sur l'origine et le développement des méthodes en géométrie' (1837), and the 'Traité de géométrie supérieure' (1852). These works, through their brilliant style, not only threw into the shade for a time the labours of contemporary German mathematicians, such as Möbius, Steiner, Plücker, and Von Staudt, but also obscured some of the single discoveries of the author himself. The 'Aperçu' was early translated into German; whereas in this country it was the Dublin school, notably Townsend and Dr Salmon, who spread a knowledge of Chasles's work.

as its leader and centre, laboured at the introduction into pure geometry of those ideas which were peculiar to the analytical method, and which gave to that method its unity, generality, and comprehensiveness. Two ideas presented themselves as requiring to be geometrically dealt with: the infinite and the imaginary—*i.e.*, the elements of a figure which lie at infinity and those which are ideal or invisible, which cannot be construed. It is usually supposed that the consideration in geometry of imaginary or invisible elements in connection with real figures in space or on the plane has been imported from algebra; but the necessity of dealing with them must have presented itself when constructive geometry ceased to consider isolated figures rigidly fixed, when it adopted the method of referring figures to each other, of looking at systems of lines and surfaces, and of moving figures about or changing them by the processes of projection and perspective. The analytical manipulations applied to an equation, which according to some system or other expressed a geometrical figure, found its counterpart in projective geometry, where, by perspective methods,—changing the centre or plane of projection,—certain elements were made to move away into infinity, or when a line that cut a circle moved away outside of it, seemingly losing its connection with it. By such devices, implying continuous motion in space, Poncelet introduced and defined points, lines, and other space elements at
35. Ideal elements. infinity, and brought in the geometrical conception of ideal and imaginary elements. "Such definitions," he says, "have the advantage of applying themselves at once to all points, lines, and surfaces whatsoever; they

are, besides, neither indifferent nor useless, they help to shorten the text and to extend the object of geometrical conceptions; lastly, they establish a point of contact, if not always real, at least imaginary, between figures which appear—*prima vista*—to have no mutual relation, and enable us to discover without trouble relations and properties which are common to them." [1] It was the principle of geometrical continuity which led Poncelet to the consideration of infinite and imaginary elements.

As we saw above, the projective methods of Poncelet had introduced into geometrical reasoning a remarkable distinction among the properties of figures. In general it was recognised that, in the methods of central and parallel projection or in drawing in perspective, certain properties or relations of the parts of a figure remain unaltered, whereas others change, become contorted or out of shape. Poncelet called the former projective or descriptive, the latter metrical, properties. This distinction introduced into all geometry since his time several most important and fundamental points of view; it divided geometrical research into two branches, which we may term positional and metrical geometry—the geometry of position and that of measurement. We know that ancient geometry started from problems of mensuration: modern geometry started, with Monge, from problems of representation or graphical description. It has thus become a habit to call ancient geometry metrical, modern geometry projective. This habit has led to an unnecessary separation of views, but in the further course of development also

[1] 'Traité des Propriétés projectives,' vol. i. p. 28.

to a unification on a higher level. But the distinction mentioned above led to another most remarkable line of thought and research which tends more and more to govern mathematical doctrine. The methods of projection are based upon the motion or upon the transformation of figures. Under such a process some relations remain unaltered or invariant, others change. As analytical methods in the hands of Plücker and others began to accommodate themselves more closely to geometrical forms, as an intimate correspondence was introduced between the figure and the formula, it became natural to study the unalterable properties of the figure in the invariant elements of the formula. This is the origin and meaning of the doctrine of Invariants.[1] It is the great merit of the English school of mathematicians, headed by Boole, Cayley, and Sylvester, both to have first conceived the idea of a doctrine of invariant

36. Invariants.

[1] "In any subject of inquiry there are certain entities, the mutual relations of which, under various conditions, it is desirable to ascertain. A certain combination of these entities may be found to have an unalterable value when the entities are submitted to certain processes or are made the subjects of certain operations. The theory of invariants in its widest scientific meaning determines these combinations, elucidates their properties, and expresses results when possible in terms of them. Many of the general principles of political science and economics can be expressed by means of invariantive relations connecting the factors which enter as entities into the special problems. The great principle of chemical science which asserts that when elementary or compound bodies combine with one another the total weight of the materials is unchanged, is another case in point. Again, in physics, a given mass of gas under the operation of varying pressure and temperature has the well-known invariant, pressure multiplied by volume and divided by absolute temperature. Examples might be multiplied. In mathematics the entities under examination may be arithmetical, algebraical, or geometrical; the processes to which they are subjected may be any of those which are met with in mathematical work. It is the principle which is valuable. It is the idea of invariance that pervades to-day all branches of mathematics." (Major P. A. MacMahon, Address, Brit. Assoc., 1901, p. 526.)

forms, and to have foreseen its importance and corresponding significance when applied to a great variety of scientific problems, notably to the projective processes in geometry. These were known to them mainly through the classical treatises of Poncelet and Chasles, the leading ideas of which had been introduced to British students by the labours of the Dublin school.[1]

The investigations referred to mark the junction of two important lines of mathematical research, which had been carried on independently in earlier times, or only united for special purposes or for the solution of special problems. The history of the progress of geometry during the nineteenth century has already shown us the use and interest which belong to two different aspects of the common object, of which the one relies mainly on processes of measurement, including number, the other mainly on processes of description, in-

[1] The history of the doctrine of invariants has been written by Dr Franz Meyer, and is published in the first volume of the 'Jahresbericht der Deutschen Mathematiker Vereinigung' (p. 79 *sqq.*) The fact that this formed the first of the several Reports which the German Mathematical Society has undertaken to publish, testifies to the great importance which belongs to this doctrine in the history of recent mathematics. A concise summary with copious references is given by the same author in the first volume of the 'Encyklopädie der Math. Wissenschaften,' p. 320 *sqq.* How necessary the form and perfection of algebraic operations was for the development of the geometrical conceptions which are laid down, *e.g.*, in the works of Plücker, can be seen in the work of Otto Hesse, who introduced elegance and conciseness into many of the expositions which, for want of this formal development, appear cumbrous in the writings of Plücker. "The analytical form in which Plücker's Researches present themselves is frequently wanting in that elegant form to which we have become accustomed, specially through Hesse. Plücker's calculations frequently bear the stamp of mere aids for representing geometrical relations. That algebraical connections possess an interest in themselves, and require an adequate representation, was realised only by a generation which habitually employed methods that had been largely devised by Plücker himself" (A. Clebsch, 'Zum Gedächtniss an Julius Plücker,' 1872, p. 8. See also Gustav Bauer, 'Gedächtnissrede auf Otto Hesse,' München, 1882).

cluding arrangement. The same difference of views can be established with regard to many other things which form the objects of other sciences. In geometry this difference obtrudes itself, as it were, in its naked form. Thus in all the natural, and even the social, sciences we have become accustomed to look first at the constituent elements or parts of things, to count and measure them, then afterwards to look at their possible arrangement, or existence together in the actual world of nature or society. Astronomy, crystallography, chemistry, geology, the natural history sciences, economics and statistics, the doctrine of chances,—all furnish, especially in their systematic development during the last hundred or hundred and fifty years, examples of the twofold aspect just referred to. The progress of these sciences, as we have abundantly seen, has depended largely upon the application of mathematical methods. As the analysis into elements or parts, and the possible synthesis of such elements in complicated structures, has become everywhere the order of study, so there must exist in the abstract science of mathematics—*i.e.*, in the framework of our scientific reasoning—not only the theory of measurement and number, but also that of combination, form or arrangement, and order.

37. **Theory of forms.** The doctrine of forms in the well-known problems of permutations and combinations begins with modern mathematics in the seventeenth century, and received scientific recognition mainly in connection with the doctrine of chances at the hands of James Bernoulli abroad, and of De Moivre in this country. The process of multiplication of binomials and polynomials leads to the formation of combinations, and

where the factors are the same, as in Newton's binomial theorem, to combinations with permutation; and consequently the doctrine of chances and of arrangements in triangular, pyramidal, or other figures is closely connected with the doctrine of series and algebraical expressions. In this country the interest in the subject has been stimulated and kept alive by isolated problems and puzzles in older popular periodicals, such as the 'Gentleman's Magazine' and the 'Ladies' Diary'; in Germany—as we noticed before—a school of mathematicians arose who attempted a systematic treatment of the whole subject, which, owing to its barrenness in practical results, brought this line of research somewhat into disrepute. What was wanted was a problem of real scientific interest and a method of abbreviation and condensation. Both were supplied from unexpected[1]

[1] The theory of arrangement or of order, also called the "Ars Combinatoria," has exerted a great fascination on some master minds, as it has also given endless opportunities for the practical ingenuity of smaller talents; among the former we must count in the first place Leibniz, and in recent times J. J. Sylvester, who conceived the "sole proper business of mathematics to be the development of the three germinal ideas—of which continuity is one, and order and number the other two" ('Philosophical Transactions,' vol. clix. p. 613). This idea has been dwelt on by Major MacMahon in his address (Brit. Assoc., 1901, p. 526), who says: "The combinatorial analysis may be described as occupying an extensive region between the algebras of discontinuous and continuous quantity. It is to a certain extent a science of enumeration, of measurement by means of integers as opposed to measurement of quantities which vary by infinitesimal increments. It is also concerned with arrangements in which differences of quality and relative position in one, two, or three dimensions are factors. Its chief problem is the formation of connecting roads between the sciences of discontinuous and continuous quantity. To enable, on the one hand, the treatment of quantities which vary *per saltum*, either in magnitude or position, by the methods of the science of continuously varying quantity and position, and, on the other hand, to reduce problems of continuity to the resources available for the management of discontinuity. These two roads of research should be regarded as penetrating deeply into the domains which they connect."

quarters—the one purely theoretical, the other practical. Accordingly the doctrine of forms and arrangements has during the last century been developed by mathematicians in two distinct interests, which only quite lately seem to approach and assist each other.

38. Theory of numbers.

The purely abstract or theoretical interest came from the side of the theory of numbers, a branch of research which was revived by Legendre in France and by the youthful genius of Gauss in Germany; the more practical one came from the theory of equations, notably in its application to problems of geometry. The methods by which these subjects were treated had in the early part of the nineteenth century undergone a great change. The older inductive method in both branches—namely, in the solution of equations and in the investigation of the properties of numbers—relied mainly on ingenious devices which were mostly of special, not of general, value. Theorems were found by induction, and had afterwards to be proved by rigorous logical deduction. Success depended on the degree of care with which the mind operated with mathematical symbols, and rested frequently on the intuition, if not the inspiration, of genius. Two of the greatest mathematical minds—Fermat[1] in France and Newton[2] in England—stood

[1] Pierre Fermat (1601-65) prepared an edition of the Treatise of Diophantus, and his marginal notes contain many theorems referring to the properties of numbers which have been the subject of much comment and examination by mathematicians of the first rank down to the present day. In letters to contemporaries he referred to many of these discoveries, and to his proofs, which he did not communicate. Some of these proofs seem not to have satisfied him, being deficient in rigour. In spite of the labours of Euler, Lagrange, Cauchy, Dirichlet, Kummer, and others, one of these theorems still awaits proof. A full account of Fermat's theorems is given in Cantor's 'Geschichte der Mathematik,' vol. ii. 2nd ed., p. 773 *sqq.* Also in W. Rouse Ball's 'History of Mathematics,' p. 260 *sqq.*

[2] Newton, in his 'Universal

foremost in having with unrivalled fertility propounded theorems which were as difficult to prove as the manner in which they had been arrived at was mysterious. The great analytical genius of Euler, who possessed unequalled resources in the solution of single problems, spent much time and power in unravelling the riddles of Fermat. In the theory of equations the general solution beyond the fourth degree baffled the greatest thinkers. The time had come when in both branches a systematic study of the properties had to be attempted. This was done for the theory of numbers by Gauss, for that of equations by Abel. Every great step in advance of this kind in mathematics is accompanied by, and dependent on, skilful abbreviations, and an easy algorithm or mathematical language. An assemblage of elements held together by the simplest operations or signs of arithmetic — namely, those of addition and multiplication—is much easier to deal with if it can be arranged with some regularity, and accordingly methods were invented by which algebraical expressions or forms were made symmetrical and homogeneous;[1] the latter property signifying that each term

39. Symmetry.

Arithmetic,' gave an interesting theorem by which the number of imaginary roots of an equation can be determined; he left no proof, and the theorem was discussed by Euler and many other writers, till at last Sylvester in 1866 found the proof of it in a more general theorem. In more recent times Jacob Steiner published a great number of theorems referring to algebraical curves (see Crelle's 'Journal,' vol. xlvii.) which have been compared by Hesse with the "riddles of Fermat." Luigi Cremona succeeded at last in proving them by a general synthetical method.

[1] The introduction of homogeneous expressions marks a great formal advance in algebra and analytical geometry. The first instance of homogeneous co-ordinates is to be found in Möbius's "Barycentric Calculus" (1826), in which he defined the position of any point in a plane by reference to three fundamental points, considering each point as the centre of gravity of those points when weighted. "The idea of co-ordinates appears here for the first time in a new

contained the same number of factors. Such forms could be written down on the pattern or model of one of their terms by simple methods of exchange or permutation of the elements. It would then not be necessary to write down all the terms but only to indicate them by their elements, these also being abbreviated by the use of indices. Rows and columns or arrangements in squares suggested themselves as easy and otherwise well-known artifices by which great masses of statistics and figures are marshalled and controlled. Out of these manifold but simple devices there grew an algebra of algebra, a symbol for denoting in a very general way symmetrical and homogeneous algebraical expressions.[1]

40. Determinants. Gauss termed such expressions Determinants: they turned up in his 'Disquisitiones Arithmeticæ' as they had done half a century before in Cramer's 'Analyse des lignes courbes algébriques.' Just as common fractions can be

garb, which soon led to a more general conception. The Barycentric co-ordinates were the first instances of homogeneous co-ordinates, . . . and already with Möbius the advantages become evident through the symmetry and elegance of his formulæ" (Hankel, 'Project. Geom.,' p. 22).

[1] Determinants were first used by Leibniz for the purpose of elimination, and described by him in a letter to the Marquis de l'Hospital (1693). The importance of his remarks was not recognised and the matter was forgotten, to be rediscovered by Cramer in the above-named work (1750, p. 657). It is interesting to note that the same difficulty of the process of elimination induced Plücker to resort to geometrical interpretation of analytical expressions, and that whilst he "saw the main advantage of his method in avoiding algebraical elimination through a geometrical consideration, Hesse showed how, through the use of Determinants, algebraical operations could receive that pliability the absence of which was the reason for Plücker to discard it." (See the account of Clebsch's work in 'Math. Ann.,' vol. vii. p. 13.) Through this invention the combinatorial analysis, which, in the hands of the school in Germany, had led into a desert, was raised again into importance. It has become still more important since the general theory of forms and of groups began to play an increasing part in modern analysis.

dealt with as if they were special things having special properties, though the latter depend only on the properties of the numbers they are made up of and their mode of connection; as powers and surds are separately examined; so the arrangements called determinants can be subjected to a special treatment, their properties ascertained, and themselves subjected to the ordinary operations of arithmetic. This doctrine, which constitutes the beginning and centre of the theory of algebraical forms or "quantics" and of algebraical operations or "tactics," was pretty fully worked out and first introduced into the course of teaching by Cauchy in France; then largely adopted by Jacobi in Germany, where Otto Hesse, trained in the ideas of Plücker, first showed its usefulness in his elegant applications to geometry. In France it was further developed by Hermite, who, together with Cayley and Sylvester in England, proclaimed the great importance of it as an instrument and as a line of mathematical thought.[1] In the latter country the idea of abbreviating and summarising algebraical operations had become quite familiar through another device which has not found equal favour abroad — namely, the Calculus of

[1] "For what is the theory of determinants? It is an algebra upon algebra; a calculus which enables us to combine and foretell the results of algebraical operations, in the same way as algebra enables us to dispense with the performance of the special operations of arithmetic. All analysis must ultimately clothe itself under this form." In this connection Sylvester ('Phil. Mag.,' 1851, Apl., p. 301) refers to Otto Hesse's "problem of reducing a cubic function of three letters to another consisting only of four terms by linear substitutions — a problem which appears to set at defiance all the processes and artifices of common algebra," as "perhaps the most remarkable indirect question to which the method of determinants has been hitherto applied."

41. Calculus of Operations.

Operations, the idea of treating algebraical operations and their symbols as quantities, and of subjecting them to arithmetical treatment separately from the material operated on. The genius of Arthur Cayley was specially fertile in this direction, as was that of Sylvester in the nomenclature or language of the doctrine of forms.[1] The merit, however, of having brought together the new ideas which emanated from the schools of Poncelet and Chasles in France, of Cayley and Sylvester in England, into a connected doctrine, and of having given the impetus to the fundamental re-

[1] The theory of invariants was gradually evolved from many independent beginnings. In 1864 Sylvester wrote ('Phil. Trans.,' p. 579), "As all roads are said to lead to Rome, so I find, in my own case at least, that all algebraical inquiries, sooner or later, end at the Capitol of Modern Algebra, over whose shining portal is inscribed the Theory of Invariants." About the same time (1863) Aronhold developed the principal ideas which lay at the foundation of the theory in organic connection and in complete generality, hereby domiciling in Germany the doctrine which had previously owed its development mainly to English, French, and Italian mathematicians (see Meyer, 'Bericht,' &c., p. 95). The different roads which Sylvester refers to can be traced, first, in the love of symbolic reasoning of Boole, who was "one of the most eminent of those who perceived that the symbols of operation could be separated from those of quantity and treated as distinct objects of calculation, his principal characteristic being perfect confidence in any result obtained by the treatment of symbols in accordance with their primary laws and conditions, and an almost unrivalled skill and power in tracing out these results" (Stanley Jevons in article "Boole," 'Ency. Brit.'); secondly, in the independent geometrical labours of Hesse in Germany (whose mathematical training combined Plücker's and Jacobi's teaching) and Dr Salmon in Dublin (who, after having transplanted Poncelet and Chasles to British soil, recognised the importance of Cayley's and Sylvester's work, and introduced in the later editions of his text-book modern algebraical methods); thirdly, in the independent investigations belonging to the theory of numbers of Eisenstein in Germany and Hermite in France. In full generality the subject was taken up and worked out by Sylvester in the 'Cambridge and Dublin Mathematical Journal' (1851-54), and by Cayley in the first seven memoirs upon Quantics (1854-61), which "in their many-sidedness, together with the exhaustive treatment of single cases, remain to the present day, for the algebraist as well as for the geometrician, a rich source of discovery" (Meyer, *loc. cit.*, p. 90).

modelling of the text-books and school-books of algebra and geometry in this country and in Germany, belongs undeniably to Dr Salmon of Dublin.[1] The conception of a form—be this geometrical or algebraic—suggests the investigation of the change, the recurrence of forms. How do forms under the process of geometrical or algebraical manipulation alter or preserve their various properties? The processes of projection practised by Monge, Poncelet, and Chasles in France had already led to a distinction between descriptive and metrical properties of geometrical figures. A corresponding examination of algebraical forms, which are all capable of geometrical representation or interpretation, would lead to the extensive and fundamental doctrine of the invariants of these forms—*i.e.*, of such arrangements of the elements as remain absolutely or proportionally unaltered during the processes of change and combination. Notably instead of the geometrical process of projection by central perspective we may employ in our algebraic formulæ a corresponding process, that which is known as linear substitution. And at the time when it was recognised that geometrical transformation had its

[1] Of Dr Salmon, whose 'Lessons introductory to the Modern Higher Algebra' appeared in 1859 (4th ed., 1855; 1st German ed. by Fiedler, 1863), Meyer says: "Recognising how the special results in this domain gradually acquired a considerable bulk, we must the more gratefully acknowledge the work of Salmon—who had already, in the direction of algebra as well as of geometry, furnished valuable contributions of his own—in undertaking the labour of collecting the widely-scattered material in a concise monograph. For the promulgation in Germany we have to thank Fiedler both for his edition of Salmon, and for having already given an independent introduction to the subject, in which especially he made Cayley's applications to projective geometry generally accessible. About the same time (1862) there appeared likewise an edition by Brioschi, which gained many adherents for the theory of Invariants in Italy."

counterpart in the transformation of algebraical forms by the processes of substitution, these latter had already been extensively studied for their own sakes in the theory of algebraical equations, which in the first quarter of the century had undergone a great development under the hands of two brilliant mathematical talents both lost to science at an early age — the Norwegian Abel and the Frenchman Évariste Galois.[1]

42. Principle of substitution.

Like all algebraical expressions, those termed equations were originally invented and commanded attention

[1] Évariste Galois is held to have been one of the greatest mathematical geniuses of modern times, who, if he had lived, might have been a rival of Abel: he was born in 1811, and died before he was twenty-one, in consequence of a duel. For a long time his writings remained unpublished and unknown, till Liouville published them in the 11th vol. of his 'Journal' (1846). Liouville was also the first to recognise the importance and absolute correctness of Galois's method, which, when submitted to the Academy in the year 1831, and reported on by Lacroix and Poisson, had appeared almost unintelligible. On the eve of his death Galois addressed a letter to his friend Auguste Chevalier, which is a unique document in mathematical literature, forming a kind of mathematical testament. He desires this letter to be published in the 'Revue Encyclopédique,' referring publicly the "importance," not the "correctness," of his discoveries to the judgment of Jacobi and Gauss, and expressing the hope that some persons would be found who would take the trouble to unravel his hieroglyphics. The first attempt to make Galois's ideas generally accessible is to be found in Serret's 'Algèbre Supérieure' (3rd ed., 1866), but it was not till after the publication of Camille Jordan's 'Théorie des Substitutions' (1870) that the short papers of Galois were recognised as containing the germs and beginnings of an entirely novel and comprehensive mathematical theory —viz., the "Theory of Groups." The relation between the writings of Abel and Galois is exhaustively treated in Prof. Sylow's Paper on Abel's work, contained in the 'Memorial Volume,' 1892, p. 24. He there says: "Le mérite de Galois ne consiste pas essentiellement dans ses propositions, mais dans la généralité de la méthode qu'il appliqua. C'est son admirable théorème fondamental qui a donné à la théorie des équations sa forme définitive, et d'où est sortie, en outre, la théorie des groupes généralisée, qui est d'une si grande importance, on peut le dire, pour toutes les branches des mathématiques, et qui déjà, entre les mains de Jordan, de Klein, de Lie, de Poincaré et d'autres, a enrichi la science d'une longue suite de découvertes importantes." The memoirs of Abel and Galois referring to the Theory of Equations have been conveniently edited, in a German translation, by H. Maser, 1889. See also Cayley's article on "Equation" in the 'Ency. Brit.,' § 32.

as instruments or devices for the solution of definite problems in arithmetic, geometry, and mechanics. The solution of the equation—*i.e.*, the expression of the unknown quantity in terms of the known quantities—served a practical end. Gradually as such solutions became more and more difficult, owing to the complexity of the formulæ, the doctrine divided itself into two distinct branches, serving two distinct interests. The first, and practically the more important one, was to devise methods by which in every single case the equations which presented themselves could be solved with sufficient accuracy or approximation; this is the doctrine of the numerical solution of equations. The other more scientific branch looked upon equations as algebraical arrangements of quantities and operations which possessed definite properties, and proposed to investigate these properties for their own sake. The question arose, How many solutions or roots an equation would admit of, and whether the expression of the unknown quantity in terms of the known quantities was or was not possible by using merely such operations as were indicated by the equation itself—*i.e.*, the common operations and the ordinary numbers of arithmetic? This doctrine of the general properties of equations received increasing attention as it became empirically known that equations beyond the fourth degree could not be solved in the most general form.[1] Why could they not be solved,

43. General solution of equations.

[1] Since the researches regarding the solubility of Equations have led on, through Galois and the French analysts, to the same line of reasoning as other researches mentioned before — viz., toward the development of the theory of groups — the history of the whole subject has aroused special interest. The earlier beginnings and the labours of forgotten analysts have been un-

and what were the conditions—*i.e.*, the special properties—of an equation which rendered it soluble? These were some of the questions which the great mathematicians, such as Gauss, Abel, and Galois, placed before themselves during the earlier part of the century. There are other unsolved problems which the nineteenth century inherited from preceding ones, where the same line of reasoning was adopted—*i.e.*, where the question was similarly reversed. Instead of trying to solve problems as yet unsolved, it was proposed to prove their general insolubility, and to show the reason of this; also to define the conditions which make a solution possible.

earthed and placed in their correct historical perspective. Prof. Burkhardt of Göttingen, to whom we also owe the chapter on this subject in the first volume of the 'Encyklopädie,' &c., contributed in the year 1892 a most interesting historical paper, "Die Anfänge der Gruppentheorie und Paolo Ruffini" ('Abhandl. zur Gesch. der Math.,' 6 Heft). In this paper he also goes back to other earlier analysts, among them Prof. Waring of Cambridge, who during his lifetime used to complain that he knew of no one who read his mathematical tracts. It appears that during nearly the last thirty years of the eighteenth century nothing had been added regarding the general theory of equations, and that Ruffini was the first to begin a new epoch in the year 1799, with the distinct assertion that a general solution of algebraic equations beyond the fourth degree, by means of radicals, was impossible, and with an attempt to prove this. His researches were therefore contemporaneous with those of Gauss, who published his 'Dissertation' (see note p. 644) in the same year, and his great arithmetical work in 1801. Although Gauss seems to have arrived at the same conclusion, and perhaps even to have anticipated much later attempts to solve the general equation of the fifth degree by other than algebraical operations (see Sylow, *loc. cit.*, p. 16), his published researches rather took the line of the study of a definite class of soluble equations which were connected with the celebrated problem of the division of the circle; a satisfactory proof of Ruffini's statement being withheld till Abel published his celebrated memoir in the year 1825 in the first volume of Crelle's 'Journal.' With this memoir the theory of equations entered a new phase, towards which the labours of Ruffini were preparatory. As in so many other cases, so also in this, the solution of the problem depended upon stricter definitions of what was meant by the solution of an equation, and by "algebraical" and other ("transcendental") functions and operations. We know that both Abel and Galois began their research by futile attempts to find a solution of the general equation of the fifth degree.

In following this altered course of investigation, an enormous amount of mathematical knowledge was gained, and problems were solved which had previously never been thought of. Especially through the theory of equations the abstract doctrine of algebraical forms was created and greatly advanced long before it was generally recognised that it had peculiar importance through the correspondence or parallelism which existed between algebraical expressions and geometrical configurations.

Out of these earlier algebraical and later combined algebraical and geometrical investigations, a novel and very useful point of view has been gradually gained which represents the most general conception of mathematical tactics. This centres in the notion of a group of elements. These elements may be quantities or operations, so that the theory of Groups embraces not only the doctrines which deal with quantities but also those which deal with arrangements and their possible changes. The older combinatorial analysis dealt mainly with assemblages of a quantity of separate elements, their number, their variety: the modern theory of groups deals rather with the processes and operations by which different arrangements can be transformed one into the other. It is an algebra of operations. The methods of transformation which presented themselves first of all were the methods known in algebra as substitution. Accordingly the first comprehensive treatise on the theory was the 'Treatise on Substitutions,' published in 1870 by M. Camille Jordan. This book forms a landmark in modern mathematics; it brought into a system 44. Theory of groups.

the beginnings of the new and comprehensive calculus of operations which were contained in the writings of Lagrange, Abel, Cauchy, and Galois, and established the terminology and the algorithm. A group of substitutions is defined as having the property that each two or more operations belonging to it and successively applied can be replaced by another single operation contained in the same group. Succeeding operations are symbolically represented by the product of two or more letters. This product has certain algebraical properties, and in analogy with common products it has factors, a degree, an index; the substitution may be cyclical and symmetric, and may have many other remarkable properties which the theory[1]

[1] The "Theory of Groups" has now grown into a very extensive doctrine which, according to the late Prof. Marius Sophus Lie (1842-99), is destined to occupy a leading and central position in the mathematical science of the future. "The conception of Group and Invariant was for him not only a methodical aspect from which he intended to review the entire older region of mathematics, but also the element which was destined to permeate and unify the whole of mathematical science" (M. Nöther, 'Math. Ann.,' vol. liii. p. 39). But though it is an undoubted fact that the largest systematic works on the subject emanate from that great Norwegian mathematician, and that his ideas have won gradual recognition, especially on the part of prominent French mathematicians, notably M. Picard ('Traité d'Analyse,' 1896, vol. iii.) and M. Poincaré, the epoch-making tract which pushed the novel conception into the foreground was Prof. F. Klein's 'Erlangen Programme' (1872), entitled "Vergleichende Betrachtungen über neuere geometrische Forschungen." To those who read and re-read this short but weighty treatise, it must indeed have been like a revelation, opening out entirely new avenues of thought into which mathematical research has been more and more guided during the last generation. The tract, which has now been translated into all the important modern languages, remained for a long time comparatively unnoticed, and, twenty years after its publication, was reprinted by the author in the 43rd volume of the 'Math. Annalen,' with some introductory remarks which indicate the changes that had taken place in the interval as regards the scope of the idea. The main result of the dissertation is this: That, primarily, for all geometrical investigations, the characteristic properties of any manifold (or arrangement) is not the element out of which it is composed, but the group, the transformations of which reveal its invariantive properties. There are, accordingly, as many different ways of

of groups investigates. Its immediate application, and the purpose for which it was elaborated, was the theory of Equations. Every equation constitutes an arrangement in which a finite number of independent elements, called constants or coefficients, is presented under a certain algebraical form. The solution of the equation means the finding of such an arrangement as when substituted in the equation for the unknown quantity, will satisfy the equation.

The conception of a group of operations standing in the defined relations is, however, capable of a great and fundamental extension into that region of mathematics which deals, not with fixed or constant, but with variable or flowing quantities; not with elements which are disconnected or discontinuous, but with such as are continuous. To understand the development of modern mathematical thought, it is accordingly necessary to go back somewhat and review the progress which the

45. Continuous and discontinuous groups.

studying any manifold (*e.g.*, such as projective geometry, line geometry, geometry of reciprocal radii, Lie's sphere geometry, analysis situs, &c.) as there are continuous groups of transformations that can be established; and there are as many invariant theories (see 'Ency. Math. Wiss.,' vol. ii. p. 402; Nöther, *loc. cit.*, p. 22). From that date onward the different kinds of groups have been defined and systematically studied, notably by Klein and Lie and their pupils. In this country, although many of the relevant ideas were contained in the writings notably of Cayley and of Sylvester, the systematic treatment of the subject was little attended to before the publication (1897) of Prof. Burnside's 'Theory of Groups of Finite Order,' and latterly of his article on the whole Theory of Groups in the 29th volume of the 'Ency. Brit.' It has been remarked by those who have studied most profoundly the development of the two great branches of mathematical tactics —viz., "The Theory of Invariants" and the "Theory of Groups"—that the progress of science would have been more rapid if the English school had taken more notice of the general comprehensive treatment by Lie, and if Lie himself had not refrained from entering more fully into the special theories of that school (see Dr F. Meyer, 'Bericht,' &c., p. 231).

conception of the variable[1] has undergone in the course of the last hundred years. Here we come upon a term which was introduced into mathematical language mainly through the writings of Euler — the term function. It is used to denote the mathematical dependence of two or more variable quantities on each

[1] To the theory of equations in algebra there corresponds the theory of differential equations in analysis; and as the theory of algebraical equations had gradually emerged in a complete form out of investigations of special equations, or sets of equations, so likewise in analysis a general theory of differential equations is gradually being evolved out of the scattered and very extensive investigations of special differential equations which presented themselves notably in the application of analysis to astronomical and physical problems. It is claimed by those who have grasped the abstract ideas of Sophus Lie, that he has taken a great step forward in the direction of a general theory of differential equations, by applying methods which suggested themselves to him through the general theory of algebraic forms and its connection with geometry. Accordingly, the theories of Lie can be termed an algebraical theory of differential equations, depending upon transformations analogous to those which had been established in the general theory of forms or quantities of which I treated above. Prof. Engel, in his obituary notice of Sophus Lie ('Deutsche Math. Ver.,' vol. viii. p. 35), tells us that in the year 1869-70, when Lie met Prof. Klein in Berlin, the former was occupied with certain partial differential equations which exhibited, under certain transformations, invariantive properties, and that Klein then pointed out "that his procedure had a certain analogy with the methods of Abel. The suggestion of this analogy became important for Lie, as he was generally intent upon following up more closely the analogies with the theory of algebraical equations." Dr H. F. Baker, in his recent article on Differential Equations in the 'Ency. Brit.' (vol. xxvii. p. 448), roughly distinguishes two methods of studying differential equations, which he names respectively "transformation theories" and "function theories," "the former concerned to reduce the algebraical relation to the fewest and simplest forms, eventually with the hope of obtaining explicit expressions of the dependent in terms of the independent variables; the latter concerned to determine what general descriptive relations among the quantities are involved by the differential equations, with as little use of algebraical calculations as may be possible." For the history of thought and connection of ideas, it is interesting to learn, through Prof. Engel, that it was not purely algebraical work,—such as is represented by Galois and Jordan, to which Lie was early introduced by Prof. Sylow,—but the study of Poncelet's and Plücker's methods which led Lie to his original conceptions, and that he was fond of calling himself a pupil of Plücker, whom he had never seen (Engel, *loc. cit.*, p. 34).

other. The question arises, What are we to understand under this term? What is a mathematical function or dependence? The question was approached by the great analysts of the second half of the eighteenth century. A preliminary answer which served the requirements of a very wide field of practical application was given by Fourier at the beginning of the nineteenth century. Since that time the question has been independently treated by two schools of Continental mathematicians. Of these the first was founded by Cauchy in France, and is mainly represented by Bernhard Riemann and his numerous pupils in Germany; the other centres in the Berlin school, headed by Weierstrass, and goes back to the work of Lagrange.

The interests which have led to this modern branch of mathematical research [1] are various, but we can 46. Theory of Functions.

[1] The literature suitable for introducing the student of mathematics to the modern theory of functions—which plays in analysis, *i.e.*, the doctrine of variable quantity, a part of similar importance to that which the theory of forms plays in algebra—is so enormous, the subject being approached from so many sides by different writers, that it seems worth while to refer to two expositions which may be read with profit, and which do not require extensive mathematical knowledge. First and foremost I would recommend Cayley's article on "Functions" in vol. ix. of the 'Ency. Brit.' Then there is the chapter on "Foundations of the General Theory of Functions," contained in the 2nd volume of the German 'Mathematical Encyclopedia,' written by Prof. Pringsheim. Cayley's article introduces the general theory after giving a short summary of the more important "known" functions, including those which presented themselves in the first half of the nineteenth century, and which I referred to in dealing with the work of Abel and Gauss (see note, p. 648). The treatment of these latter functions, which had been brought to a certain degree of perfection by Jacobi, had made it evident that more general aspects had to be gained and broader foundations laid. But ever since the middle of the eighteenth century another development of mathematical ideas had been going on which started from the solution of a problem in mathematical physics—namely, that of vibrating strings, which led in the sequel to

distinguish two which are very prominent, and are roughly represented by the two schools just referred to. In the first place, a function can be formally defined as an assemblage of mathematical symbols, each of which denotes a definite operation on one or more quantities. These operations are partly direct, like addition, multiplication, &c.; partly indirect or inverse, like subtraction, division, &c. Now, so far as the latter are concerned, they are not generally and necessarily practicable, and the question arises, When are they practicable, and if they are not, what meaning can we connect with the mathematical symbol? In this way we arrive at definitions for mathematical functions which cannot immediately be reduced to the primary operations of arithmetic, but which form special expressions that become objects of research as to their properties and as to the relation they bear to those fundamental operations upon which all our methods of calculation depend. The inverse operations, represented by negative, irrational, and imaginary quantities; further, the operations of integration in its definition as the in-

a certain finality when Fourier introduced his well-known series and integrals, by which any kind of functionality or mathematical dependence, such as physical processes seem to indicate, could be expressed. The work of Fourier, which thus gave, as it were, a sort of preliminary specification under which a large number of problems in physical mathematics could be attacked and practically solved, together with the stricter definitions introduced by Lejeune Dirichlet, settled for a time and for practical purposes the lengthy discussions which had begun with Euler, Daniel Bernoulli, d'Alembert, and Lagrange. The above-named chapter, written by Prof. Pringsheim, gives an introduction to the subject showing the historical genesis of the conception of function and the various changes it was subjected to, and then proceeds to expositions and definitions mostly taken from the lectures of Weierstrass (see p. 8), whereas Cayley's article introduces us to the elements of the general theory of functions as they were first laid down by Riemann in the manner now commonly accepted.

verse of differentiation,—led early to investigations of the kind just mentioned. The experience that ordinary fractions might be expressed by decimal fractions—*i.e.*, by finite or infinite series—led to the inverse problem of finding the sum of such series and many other answerable and apparently unanswerable problems. The older method of research consisted in treating these problems when and as they arose: new chapters were accordingly added to the existing chapters of the text-books, dealing with special functions or mathematical expressions. It was only towards the end of the eighteenth century, and at the beginning of the nineteenth, that Lagrange, Gauss, and Cauchy felt and proclaimed the necessity of attacking the question generally and systematically; the labours of Euler having accumulated an enormous mass of analytical knowledge, a great array of useful formulæ, and amongst them not a few paradoxes which demanded special attention. I have already had occasion to refer to the problem of the general solution of equations as an instance where, in the hands of Abel, the tentative and highly ingenious attempts of earlier analysts were replaced by a methodical and general treatment of the whole question. Another chapter of higher mathematics, the investigation of expressions which presented themselves in the problems of finding the length of the arc of an ellipse, and which opened the view into the large province of the so-called higher transcendents, gave Abel further occasion of laying new foundations and of creating a general theory of equations or of forms.

But yet another interest operated powerfully in the

direction of promoting these seemingly abstract researches. Nature herself exhibits to us measurable and observable quantities in definite mathematical dependence;[1] the conception of a function is suggested by all the processes of nature where we observe natural phenomena varying according to distance or to time.

47. Physical analogies.

[1] Nearly all the "known" functions have presented themselves in the attempt to solve geometrical, mechanical, or physical problems, such as finding the length of the arc of the ellipse (elliptic functions); or answering questions in the theory of attraction (the potential function and other functions, such as the functions of Legendre, Laplace, and Bessel, all comprised under the general term of "harmonic functions"). These functions, being of special importance in mathematical physics, were treated independently before a general theory of functions was thought of. Many important properties were established, and methods for the numerical evaluation were devised. In the course of these researches other functions occurred, such as Euler's "Gamma" function and Jacobi's "Theta" function, which possessed interesting analytical properties. These functions, suggested directly or indirectly by applications of analysis, did not always present themselves in a form which indicated definite analytical processes, such as processes of integration or the summation of series. Very frequently they presented themselves, not in an "explicit" but in an "implicit" form; their properties being expressed by certain conditions which they had to fulfil. It then remained a question whether a definite symbol, indicating a set of analytical operations, could be found. This arises from the fact that the solution of most problems in mechanics and physics starts from the assumption that, though the finite observable phenomena of nature are extremely intricate, they are, nevertheless, compounded out of comparatively simple elementary processes, which take place between the discrete atoms, or the elementary but continuous portions of matter. Mathematically expressed, this means that the relations in question present themselves in the form of differential equations, and that the solution of them consists in finding functions of finite (observable) quantities which satisfy the special conditions. A comparatively small number of differential equations has thus been found empirically to embrace very large and apparently widely separated classes of physical phenomena, suggesting physical relations between those phenomena which might otherwise have remained unnoticed. The physicist or astronomer thus hands over his problems to the mathematician, who has either to integrate the differential equations, or, where this is not possible, at least to infer the properties of the functions which would satisfy them—in fact, the differential equation becomes a definition of the function or mathematical relation. In consequence of this the theory of differential equations is, as Sophus Lie has said, by far the most important branch of mathematics.

The attraction of the heavenly bodies varies with the distance, the velocity of a falling stone or the cooling of a hot body varies with the interval of time which has lapsed or flown. We are now so much accustomed to represent such dependence by curves drawn on paper, that we hardly realise the great step in advance towards definiteness and intelligibility that this device marks in all natural sciences and in many practical pursuits. But the representation of the natural connections of varying quantities by curves also forms the connecting link with the other class of researches just mentioned. Descartes had shown how to represent algebraical formulæ by curves in the plane and in space; and at the beginning of the nineteenth century this method was modified by Gauss and Cauchy so as to deal also with the extended conception of number which embraced the imaginary unit. Two questions arise, Is it possible to represent every arbitrary dependence such as we meet with in the graphical description of natural phenomena by a mathematical formula—*i.e.*, by a formula denoting several specified mathematical operations in well-defined connections? and the inverse question, Is it possible to represent every well-defined arrangement of symbols denoting special mathematical operations graphically by curves in the plane or in space? The former question is one of vital importance in the progress of astronomy, physics, chemistry, and many other sciences, and has accordingly occupied many eminent analysts ever since Fourier gave the first approximative answer in his well-known series: the latter question can only be answered by much stricter defini-

tions of all the more advanced and of some even of the elementary operations which analysts had become accustomed to use without a previous knowledge of the range of their validity. All applications of mathematics consist in extending the empirical knowledge which we possess of a limited number or region of accessible phenomena into the region of the unknown and inaccessible; and much of the progress of pure analysis consists in inventing definite conceptions, marked by symbols, of complicated operations; in ascertaining their properties as independent objects of research; and in extending their meaning beyond the limits they were originally invented for,—thus opening out new and larger regions of thought.

48. The potential. A brilliant and most suggestive example of this kind of reasoning was afforded by a novel mode of treating a large class of physical problems by means of the introduction of a special mathematical function, termed by George Green, and later by Gauss, the "Potential" or "Potential function."[1] All the problems of Newtonian attraction were concentrated in the study of this formula: and when the experiments of Coulomb and Ampère showed the analogy that existed between electric and magnetic forces on the

[1] See vol. i. p. 231 of this work. The history of the subject has been written by Todhunter ('History of the Theories of Attraction and the Figure of the Earth,' 2 vols., 1873) for the earlier period down to 1832. For the later period see Bacharach's 'Abriss der Geschichte der Potentialtheorie,' Göttingen, 1883; for the connection of the theory with Riemann's mathematical methods, especially Prof. F. Klein's tract, 'Ueber Riemann's Theorie der algebraischen Functionen' (Leipzig, 1882, trans. by F. Hardcastle, Cambridge, 1893); Prof. Carl Neumann's 'Untersuchungen über das Logarithmische und Newtonische Potential' (Leipzig, 1877); Dr Burkhardt's 'Memorial Lecture on Riemann' (Göttingen, 1892); and jointly with Dr Franz Meyer, the same author's chapter on "Potentialtheorie" in the 2nd volume (p. 464) of the 'Encyclopädie der Math. Wiss.,' 1900.

one side, and Newtonian forces on the other; still more when Fourier, Lamé, and Thomson (Lord Kelvin) pointed to the further analogy which existed between the distribution of temperature in the stationary flow of heat and that of statical electricity on a conductor, and extended the analogy to hydrostatics and hydrodynamics, —it became evident that nature herself pointed here to a mathematical dependence of the highest interest and value. Many eminent thinkers devoted themselves to the study of this subject, but it was reserved for Bernhard Riemann to generalise the mode of reasoning peculiar to these researches into a fundamentally novel method for the explanation and definition of mathematical function or dependence.[1]

[1] Although Riemann's original method of dealing in a general way with algebraical functions is here introduced as a generalisation of certain ideas suggested by mathematical physics, it was not in this way that they were introduced to the mathematical world. This was done in his very abstract and difficult memoir, 'Theorie der Abel'schen Functionen' (published in 1857 in vol. liv. of Crelle's 'Journal'). In this memoir the connection which existed with mathematical physics was not patent, and it took a long time before his methods, which seemed to be a development of Cauchy's earlier researches, were understood and fully appreciated. It was only after he had lectured repeatedly on the subject, and initiated a number of younger mathematicians, who now occupy many of the chairs at the German universities, that the discoveries and inventions of Riemann received their deserved appreciation. Even in his own lectures on mathematical physics—notably on partial differential equations (including harmonics) and the theory of the potential—he did not lead up to the fundamental ideas which he developed in his lectures on the theory of the Abelian functions. Some light is thrown on the subject of the genesis of Riemann's ideas by his dissertation written in the year 1851, though even the biographical notice attached to the 1st edition of his works (1876) did not deal with the origins of his theory. It seems, therefore, correct to date the adequate recognition of Riemann's work in wider circles from the publication in 1882 of Prof. F. Klein's tract mentioned above. Like several other short treatises of this eminent living mathematician, it must have thrown quite a new light upon the subject; and, like several of his other writings, it revealed connections between regions of thought which to many students must have appeared isolated. "Through the treatment initiated by Klein, the theory of

49. Riemann. The peculiarity of such dependence, as exemplified in the phenomena of the steady flow of heat or of electric distribution, consisted in this, that if at certain points or in certain regions of space the thermal or electrical conditions were defined and known by actual observation, then the whole distribution in other points and regions was completely determined. Those boundary conditions could therefore be regarded as the necessary and sufficient definition of the whole existing distribution. Translated into mathematical language, this means that functions exist which are completely defined by boundary values and singularities—*i.e.*, values at single points. Nature herself had shown the way to define and calculate measured relations when through their intricacy they evaded the grasp of the ordinary operations of algebra.[1] Plücker had already in geometry (following in the lines of Newton), when attacking the problem of the infinite variety of higher curves, suggested the method of classifying them according to their characteristic properties or singularities. What had been done by geometers and physicists in isolated cases with the expenditure of much ingenuity and skill, Riemann and his school elevated to the rank of a general method and doctrine.

functions acquires a great degree of clearness and connectedness, which is mainly gained by conceptions derived from the (physical) theory of the potential, and thus exhibits the intimate relationship of these theories" (Bacharach, 'Geschichte der Potentialtheorie,' Göttingen, 1883, p. 71).

[1] On this subject see Burkhardt's 'Memorial Lecture on Riemann' (Göttingen, 1892), p. 5, &c.; Bacharach (*loc. cit.*), p. 30, &c. The latter especially with reference to the theorem called by Clerk-Maxwell "Thomson's theorem" ('Cambridge and Dublin Mathematical Journal,' 1848, or 'Reprint of Papers on Electro-statics,' &c., p. 139); and abroad 'Dirichlet's Principle,' after Riemann (1857). Further, Brill and Nöther's "Bericht" ('Math. Ver.,' vol. iii. p. 247); and lastly, a very suggestive address by Prof. Klein ("On Riemann's Influence on Modern Mathematics") to the meeting of the German Association in Vienna in 1894 ('Report,' p. 61).

It is a process of generalisation and simplification. Moreover, Riemann's manner of proceeding brought with it the gain that he could at once make the various theorems of the doctrine of the potential useful for purely mathematical purposes: the equation which defined the potential in physics became the definition of a function in mathematics.[1]

[1] "One may define Riemann's developments briefly thus: that, beginning with certain differential equations which the functions of the complex variable satisfy, he is enabled to apply the principles of the potential theory. His starting-point, accordingly, lies in the province of mathematical physics" (Klein, 'Vienna Report,' *loc. cit.*, p. 60). By starting with physical analogies Prof. Klein evades certain difficulties which the purely mathematical treatment had to encounter. In the preface to his tract of the year 1882, quoted above,—in introducing his method of explaining Riemann's theory,—he says: "I have not hesitated to make exactly these physical conceptions the starting-point of my exposition. Instead of them, Riemann, as is well known, makes use in his writings of Dirichlet's principle. But I cannot doubt that he started from those physical problems, and only afterwards substituted Dirichlet's principle in order to support the physical evidence by mathematical reasoning. Whoever understands clearly the surroundings among which Riemann worked at Göttingen, whoever follows up Riemann's speculations as they have been handed down to us, partly in fragments, will, I think, share my opinion." And elsewhere he says: "We regard as a specific performance of Riemann in this connection the tendency to give to the theory of the potential a fundamental importance for the whole of mathematics, and further a series of geometrical constructions or, as I would rather say, of geometrical inventions" ('Vienna Report,' p. 61). Klein then refers to the representation on the so-called "Riemann surface," which is historically connected, as Riemann himself points out, with the problem which Gauss first attacked in a general way—viz., the representation of one surface on another in such a manner that the smallest portions of the one surface are similar to those of the other: a problem which is of importance in the drawing of maps, and of which we possess two well-known examples in the stereographic projection of Ptolemy and the projection of Mercator. This method of representation was called by Gauss the "Conformal Image or Representation." His investigations on this matter were suggested by the Geodetic Survey of the kingdom of Hanover, with which he was occupied during the years 1818 to 1830. (See Gauss, 'Werke,' vol. iv., also his correspondence with Schumacher and Bessel.) A very complete treatise on this aspect of Riemann's inventions is that by Dr J. Holtzmüller, 'Theorie der Isogonalen Verwandschaften' (Leipzig, 1882). On the historical antecedents of Riemann's conception, which for a long time appeared somewhat strange, not to say artificial, see Brill and Nöther's frequently quoted "Report" ('Bericht der Math. Verein.,' vol. iii.), p. 256 *sqq.*

In the investigation of those higher functions which the purely analytical methods of Abel and his followers had forced upon the attention of mathematicians, the methods of Riemann proved to be eminently useful and suggestive. But these novel methods themselves had been imported into the pure science from the side of its application in physics. The value of such ideas has always been questioned by another class of thinkers who aim at building up the edifice of the science by rigorous logic, without making use of practical devices which could only be legitimately employed when once their validity had been thoroughly proved and its limits defined. The merit of having done this in the whole domain of those conceptions which, since the age of Descartes, Newton, and Leibniz, had been introduced as it were from the outside into analysis, belongs to the school of mathematicians headed in Germany by Karl Weierstrass.

50. Weierstrass.

Riemann had grown up in the traditions of the school of mathematical thought which was inspired by Gauss and Weber in Göttingen. Geometrical representation and physical application, including the immediate evidence of the senses, formed a large and important factor in the body of arguments by which scientific discovery and invention was carried on in that school; though Gauss himself made logical rigour the final test of maturity in all his published writings, abstaining in many cases from communicating his results when they had not satisfactorily passed that test in his own mind. Through this self-imposed restriction he had permitted important discoveries, which led to large increase of mathematical knowledge, to be anticipated by others.

The cases of Cauchy, Abel, and Jacobi are the best-known instances. Through their labours an entirely new field had been prospected and partially cultivated. It was to this that Weierstrass, the other great leader in modern theory, was attracted. He made the clear definition and logical coherence of the novel conceptions which it involved his principal aim. Gauss had laboured without assistance at similar problems, making many beginnings which even his colossal intellect could not adequately develop. Weierstrass early gathered around him a circle of ardent and receptive pupils and admirers,[1] to whose care and detailed elaboration he

[1] The researches of Weierstrass (1815 to 1897) began somewhat earlier than those of Riemann, but only became generally known and appreciated in their fundamental originality through his pupils—his academic influence dating from the year 1861. Some account of Weierstrass's activity is given by Emil Lampe in the 6th volume (1899) of the 'Bericht der Math. Verein.,' p. 27, &c. The genesis of his ideas is traced by Brill and Nöther in the Report quoted in the last note, and by M. Poincaré in 'Acta Math.,' vol. xxii. The former divides his Researches roughly into two periods, during the first of which (1848-56) he dealt with what Cayley would call "known" functions; progress during this period depending not so much upon fundamentally new ideas as upon an investigation of special problems and great analytical skill. The second period begins in the year 1869, and is devoted to nothing less than the building up of the entire structure of mathematical thought from the very beginning upon altered definitions, through which the dilemmas and paradoxes would be obviated that had shown themselves ever since the middle of the eighteenth century in consequence of a too confident application and extension of conventional ideas suggested mainly by practical problems. The elements of this grand edifice are now largely accepted, not only in Germany, but also in France, Italy, and England. In Germany Prof. O. Stolz, through his works on General Arithmetic, 2 vols. (1885 and 1886), and the Calculus, 3 vols. (1893 to 1899), has probably done more than any other academic teacher to utilise the new system of mathematical thought for the elementary course of teaching. It seems of importance to state, however, that outside of the circle of Weierstrass's influence, and quite within the precincts of Riemann's school, the necessity was felt of strengthening the foundations on which research in higher mathematics was carried on, by going back to the fundamental ideas of arithmetic. The principal representative of this line of research was Hermann Hankel (1839-73), a pupil of Riemann's, who, in the

confided many separate and lengthy investigations. It was through one of these that a test-case, in which existing mathematical definitions broke down, was published in 1872. It forms a kind of era in the history of

middle of the sixties, delivered lectures at the University of Leipsic upon "Complex numbers and their functions," starting in a characteristic manner with that extended algebra which Cauchy and Riemann had used to such good purpose. The first part of these lectures was published in 1867. In the preface Hankel says: "In the natural sciences we witness in recent times the distinct tendency to ascend from the world of empirical detail to the great principles which govern everything special and connect it into a whole—*i.e.*, the desire for a philosophy of nature, not forced upon us from outside, but naturally evolved out of the subject itself. Also in the domain of mathematics a similar want seems to make itself generally felt—a want which has always been alive in England." Had the author not been prematurely taken away, there is no doubt that he would have still more largely contributed to the revolution of mathematical ideas now in progress. As it is, he made one further important contribution, of which more hereafter. In Italy Prof. Ulisse Dini began to lecture in the year 1871 to 1872 on the theory of functions, and published his lectures in 1878. A translation was brought out in German (1892) by Prof. Lüroth and Mr A. Schepp, in which many of the modern developments are utilised. In France we owe to M. Jules Tannery a valuable introduction to the theory of functions of one variable, based upon a series of lectures delivered in the École Normale in 1883, in which, as he says (Preface, p. vii), he collected the labours of Cauchy, Abel, Lejeune Dirichlet, Riemann, Ossian Bonnet, Heine, Weierstrass, and others; after which he considers that nothing essential need be added in the way of elucidation of the foundations of the theory. M. Émil Borel published in 1898 'Lectures on the Theory of Functions,' the first of a series of text-books dealing with various aspects of the theory of functions, in which he largely refers to the labours of Weierstrass. Before Weierstrass's theory had become known, however, M. Méray had already entered upon an exposition of the foundations of analysis on lines which had much analogy with those adopted by Weierstrass. In England the late Prof. Clifford had occupied himself in various memoirs with the theories of Riemann; but we owe the first comprehensive treatise, embracing the work of Riemann as well as that of Weierstrass, to Prof. Forsyth ('Theory of Functions of a Complex Variable,' Cambridge, 1893). Almost simultaneously Professors Harkness and Morley published a 'Treatise on the Theory of Functions,' and in 1898 an 'Introduction to the Theory of Analytic Functions,' in which they in the main adopted the point of view of Weierstrass. A very original thinker, whose independent researches reach back to the year 1872, and who played an important part in the investigation of many obscure points, was the late Prof. Paul Du Bois-Reymond, who published in 1882 the first part of his 'Allgemeine Functionentheorie,' containing the

mathematical thought. Up to that time "one would have said that a continuous function is essentially capable of being represented by a curve, and that a curve has always a tangent. Such reasoning has no mathematical value whatever; it is founded on intuition, or rather on a visible representation. But such representation is crude and misleading. We think we can figure to ourselves a curve without thickness; but we only figure a stroke of small thickness. In like manner we see the tangent as a straight band of small thickness, and when we say that it touches the curve, we wish merely to say that these two bands coincide without crossing. If that is what we call a curve and a tangent, it is clear that every curve has a tangent; but this has nothing to do with the theory of functions. We see to what error we are led by a foolish confidence in what we take to be visual evidence. By the discovery of this striking example Weierstrass has accordingly given us a useful reminder, and has taught us better to appreciate the faultless and purely arithmetical methods with which he more than any one has enriched our science."[1]

"metaphysics and theory of the fundamental conceptions in mathematics: quantity, limit, argument, and function" (Tübingen). This work touches the borderland of mathematics and philosophy, as does the same author's posthumous work 'Uber die Grundlagen der Erkenntniss in den exacten Wissenschaften' (Tübingen, 1890), and will occupy us in another place.

[1] M. Poincaré in the 'Acta Mathematica,' vol. xxii., "L'œuvre mathématique de Weierstrass," p. 5. The "test-case" referred to in the text consisted in the publication by Weierstrass (in the year 1872, 'Trans. Berlin Academy,' reprinted in Weierstrass's 'Math. Werke,' vol. ii. p. 71) of the proof of the existence of a continuous function which nowhere possessed a definite (finite or infinite) differential coefficient. This example cleared up a point brought into prominence by Riemann in his posthumously (1867) published Inaugural Dissertation of 1854 ('Werke,' p. 213). The question had already, following on Riemann's suggestions, been discussed by Hermann Hankel in a

Before Weierstrass, Cauchy and Riemann had attempted to define the vague term "function" or mathematical dependence. Both clung to the graphical representation so common and so helpful in analysis since Descartes invented it. We have, of course, in abstract science, a right to begin with any definition we choose. Only the definition must be such that it

remarkable tract on "Oscillating functions," in which he drew attention to the existence of functions which admit of an integral, but where the existence of a differential coefficient remains doubtful. In fact, it appears that the question as to the latter had never been raised; the only attempt in this direction being that of Ampère in 1806, which failed (Hankel, p. 7). Hankel in his original investigation showed that a continuous curve might be supposed to be generated by the motion of a point which oscillated to and fro, these oscillations at the limit becoming infinitely numerous and infinitely small: a curve thus generated would present what he called "a condensation of singularities" at every point, but would possess no definite direction, hence also no differential coefficient. The arguments and illustrations of Hankel have been criticised and found fault with. He nevertheless deserves the credit of having among the first attempted "to gain a firm footing on a slippery road which had only been rarely trodden" (p. 8). In this tract (which is reprinted in 'Math. Ann.,' vol. xx.), as well as in his valuable article on "Limit" (Ersch und Gruber, 'Encyk.,' vol. xc. p. 185, art. "Grenze"), Hankel did much to establish clearly the essential point on which depends the entire modern revolution in our ideas regarding the foundations of the so-called infinitesimal calculus; reverting to the idea of a "limit," both in the definition of the derived function (limit of a ratio) and of the integral (limit of a sum) as contained in the writings both of Newton and Leibniz, but obscured by the method of "Fluxions" of the former and the method of "Infinitesimals" of the latter. Lagrange and Cauchy had begun this revolution, but it was not consistently and generally carried through till the researches of Riemann, Hankel, Weierstrass, and others made rigorous definitions necessary and generally accepted. It is, however, well to note that in this country A. de Morgan very early expressed clear views on this subject. Prof. Voss, in his excellent chapter on the Differential and Integral Calculus ('Encyk. Math. Wiss.,' vol. ii. i. p. 54, &c.), calls the later period the period of the purely arithmetical examination of infinitesimal conceptions, and says (p. 60), "The purely arithmetical definition of the infinitesimal operations which is characteristic of the present critical period of mathematics has shown that most of the theorems established by older researches, which aimed at a formal extension of method, only possess a validity limited by very definite assumptions." Such assumptions were tacitly made by earlier writers, but not explicitly stated.

corresponds with conditions which we meet with in reality, say in geometry and physics, otherwise our science becomes useless: further, our definitions must be consistent, and follow logically from the fundamental principles of arithmetic, otherwise we run the risk of sooner or later committing mistakes and encountering paradoxes. We have two interests to serve: the extension of our knowledge of functions and the rigorous proof of our theorems. The methods of Riemann and of Weierstrass are complementary. "By the instrument of Riemann we see at a glance the general aspect of things—like a traveller who is examining from the peak of a mountain the topography of the plain which he is going to visit, and is finding his bearings. By the instruments of Weierstrass analysis will, in due course, throw light into every corner, and make absolute clearness shine forth."[1] The complementary character of

51. Riemann and Weierstrass compared.

[1] Poincaré, *loc. cit.*, p. 7. Similarly Prof. Klein (*loc. cit.*, 'Vienna Report,' p. 60): "The founder of the theory [viz., of functions] is the great French mathematician Cauchy, but only in Germany has it received that modern stamp through which it has, so to speak, been pushed into the centre of our mathematical convictions. This is the result of the simultaneous exertions of two workers—Riemann on the one side and Weierstrass on the other. Although directed to the same end, the methods of these two mathematicians are in detail as different as possible: they almost seem to contradict each other, which contradiction, viewed from a higher aspect, naturally leads to this—that they mutually supplement each other. Weierstrass defines the functions of a complex variable analytically by a common formula — viz., the 'Infinite Power Series'; in the sequel he avoids geometrical means as much as possible, and sees his specific aim in the rigour of proof. Riemann, on the other side, begins with certain differential equations. The subject then immediately acquires a physical aspect. . . . His starting-point lies in the region of mathematical physics." We now know from the biographical notice of Riemann, attached to his collected works (1st ed., p. 520), that he was pressed (in 1856) by his mathematical friends to publish a *résumé* of his Researches on Abelian functions—"be it ever so crude." The reason was that Weierstrass was already at work on the same subject. In consequence of Riemann's

the labours of the two great analysts is nowhere better shown than in the special manner in which Weierstrass succeeded in strengthening the foundations[1] on which much of Riemann's work rests.

The labours of the great analysts—Gauss, Cauchy, Riemann, and Weierstrass—all tended to increase our

publication Weierstrass withdrew from the press an extensive memoir which he had presented in the year 1857 to the Berlin Academy, because, as he himself says (Weierstrass, 'Math. Werke,' vol. iv. p. 10): "Riemann published a memoir on the same problem which rested on entirely different foundations from mine, and did not immediately reveal that in its results it agreed completely with my own. The proof of this required investigations which were not quite easy, and took much time; after this difficulty had been removed a radical remodelling of my dissertation seemed necessary," &c., &c. The mutual influence of Riemann's and Weierstrass's work is also referred to by Weierstrass in a letter to Prof. Schwarz, dated 1875, in which he utters what he calls his confession of faith: "The more I ponder over the principles of the theory of functions — and I do this incessantly — the stronger grows my conviction that it must be built up on the foundation of algebraical truths, and that, therefore, to employ for the truth of simple and fundamental algebraical theorems the 'transcendental,' if I may say so, is not the correct way, however enticing *prima vista* the considerations may be by which Riemann has discovered many of the most important properties of algebraical functions. It is a matter of course that every road must be open to the searcher as long as he seeks; it is only a question of the systematic demonstration" (Weierstrass, 'Werke,' vol. ii. p. 235).

[1] This refers mainly to Weierstrass's investigation of the principle called by Riemann "Dirichlet's principle," but which had been stated already with great generality by Thomson (Lord Kelvin) in the year 1847. The validity of this method depended on a certain minimum theorem. Weierstrass has shown that the existence of such a minimum is not evident, and that the argument used is not conclusive. He laid before the Berlin Academy, in the year 1870, a communication giving a test-case to prove that Dirichlet's method was not generally valid ('Werke,' vol. ii. p. 49). "Through this," Prof. Klein says (*loc. cit.*, p. 67), "a great part of Riemann's developments become invalidated. Nevertheless the far-reaching results which Riemann bases upon the principle are all correct, as was shown later on exhaustively and with all rigour by Carl Neumann and H. A. Schwarz. Indeed we must come to the conclusion that Riemann himself arrived at these theorems by a physical intuition, and only afterwards resorted to the principle referred to in order to have a consistent mathematical line of reasoning" (*loc. cit.*, p. 67). See on this also Poincaré (*loc. cit.*, pp. 10 and 15), who gives other instances where the work of Weierstrass supported that of Riemann.

knowledge of the higher mathematical relations, but also to reveal the uncertainty and absence of rigorous definition of the foundations of arithmetic and of geometry. Accordingly we find these great thinkers continually interrupting their more advanced researches by examinations of the principles. This feeling of uncertainty had led, ever since the end of the eighteenth century, to many isolated attacks and half-philosophical discussions by various writers in this country and abroad. Many of them remained long unrecognised; such were the suggestive writings of Hamilton, De Morgan, Peacock in England, Bolzano[1] in Bohemia,

52. Examination of foundations.

[1] The merits of Bernhard Bolzano (1781-1848) as one of the earliest representatives of the critical period of mathematics were recognised after a long interval of neglect by Hankel in his article on "Limit" mentioned above. This philosophical mathematician published many years before Cauchy a tract on the Binomial Theorem (Prague, 1816), in which he gives, in Hankel's opinion, the first rigid deduction of various algebraical series. "Bolzano's notions as to convergency of series are eminently clear and correct, and no fault can be found with his development of those series for a *real* argument (which he everywhere presupposes); in the preface he gives a pertinent criticism of earlier developments of the Binomial Theorem, and of the unrestricted use of infinite series, which was then common. In fact, he has everything that can place him in this respect on the same level with Cauchy, only not the art peculiar to the French of refining their ideas and communicating them in the most appropriate and taking manner. So it came about that Bolzano remained unknown and was soon forgotten; Cauchy was the happy one who was praised as a reformer of the science, and whose elegant writings were soon widely circulated." (Hankel, *loc. cit.*, p. 210.) Following on this statement of Hankel and a remark of Prof. H. A. Schwarz, who looks upon Bolzano as the inventor of a line of reasoning further developed by Weierstrass ('Journal für Mathematik,' vol. lxxiv. p. 22, 1872), Prof. O. Stolz published in 1881 ('Math. Ann.,' vol. xviii. p. 255) an account of the several writings of Bolzano, beginning in the year 1810, in so far as they referred to the principles of the Calculus. "All these writings are remarkable inasmuch as they start with an unbiassed and acute criticism of the contributions of the older literature" (*loc. cit.*, p. 257). A posthumous tract by Bolzano, 'Paradoxieen des Unendlichen,' was republished in 1889 in 'Wissenschaftliche Classiker,' vol. ii., Berlin (Meyer and Müller). As stated above, Hankel was also one of the first to draw attention to the originality and importance of Hermann Grassmann's work.

Bolyai in Hungary, Lobatchevski in Kasan, Grassmann in Stettin. Most of these were unknown to each other. However, near the beginning of the last third of the century three distinct publications created a great stir in the mathematical world, brought many scattered but cognate lines of reasoning together, and made them mutually fertile and suggestive. These three were—*first*, the publication in 1860 of Gauss's correspondence with Schumacher, in which two letters of the former, dated May and July 1831,[1] became known, where he referred to his extensive but unwritten and unfinished speculations on the foundations of geometry and the theorem which refers to the sum of the angles in a triangle. The *second* was the publication in 1867 of the first and only part of Hermann Hankel's "Lectures on the Complex Numbers and their Functions."[2] The *third* was the posthumous publication in the same year of Riemann's paper, dated 1854,[3] "On the Hypotheses which lie at the Foundation of Geometry." Almost simultaneously there appeared the first of Helmholtz's two important papers[4] on the

[1] See 'Briefwechsel zwischen Gauss und Schumacher,' ed. Peters, 1860, vol. ii. pp. 260, 268.

[2] The small volume contains so much original and historical matter that I have on several occasions referred to it. See above, pp. 645, 653.

[3] Riemann, 'Math. Werke,' 1st ed., p. 254 *sqq.*

[4] The first publication of Helmholtz was a lecture on "the actual foundations of geometry," which he delivered on the 22nd May 1868 to the Medical Society at Heidelberg. This communication, which referred to investigations carried on for many years,—notably in connection with the theory of the colour-manifold,—was occasioned by the publication of Riemann's paper in the 'Transactions' of the Göttingen Society. He had heard of this through Schering, to whom he wrote on the 21st April 1868 before having seen Riemann's paper: "I have myself been occupied with the same subject during the last two years, in connection with my researches in physiological optics. . . . I now see, from the few hints which you give as to the

same subject, through which it became more widely known and attracted the attention of other than purely mathematical writers. The small but eminently suggestive volume of Hankel showed the necessity of a revision and extension of the fundamental principles and definitions [1] of general arithmetic and algebra as

result of the investigation, that Riemann has arrived at exactly the same results. My starting-point was the question, How must a magnitude of several dimensions be constituted, if solid bodies are to move in it everywhere continuously, monodromically, and as freely as bodies move in real space?" On receiving from Schering a reply with a copy of Riemann's paper, Helmholtz wrote (18th May), "I enclose a short exposition of that which in my researches on the same subject is not covered by Riemann's work." A fuller paper, with the title "On the Facts which lie at the foundation of Geometry," appeared in the 'Göttinger Nachrichten,' June 3, 1868. See Helmholtz, 'Wiss. Abhandl.,' vol. ii. pp. 610 and 618, &c.; also 'H. von Helmholtz,' by Leo Koenigsberger (1903), vol. ii. p. 138, &c. In another lecture, "On the origin and meaning of the Axioms of Geometry" (1870, reprinted in abstract in 'The Academy,' vol. i.), as well as in an article in vol. i. of 'Mind' (p. 301), he discussed "the philosophical bearing of recent inquiries concerning geometrical axioms and the possibility of working out analytically other systems of geometry with other axioms than Euclid's" (reprinted in vol. ii. of 'Vorträge und Reden').

[1] In this treatise Hankel introduced into German literature the three terms "distributive," "associative," and "commutative" to define the three principles which govern the elementary operations of arithmetic, and introduced further what he calls the principle of the permanence of formal rules in the following statement: "If two forms, expressed in the general terms of universal arithmetic, are equal to each other, they are to remain equal if the symbols cease to denote simple quantities; hence also if the operations receive a different meaning." Hankel seems to have been led to his definitions by a study of French and English writers, among whom he mentions Servois ('Gergonne's Ann.,' v. p. 93, 1814) as having introduced the terms "distributive" and "commutative," and Sir W. R. Hamilton as having introduced the term "associative." He further says (p. 15): "In England, where investigations into the fundamental principles of mathematics have always been treated with favour, and where even the greatest mathematicians have not shunned the treatment of them in learned dissertations, we must name George Peacock of Cambridge as the one who first recognised emphatically the need of formal mathematics. In his interesting report on certain branches of analysis, the principle of permanence is laid down, though too narrowly, and also without the necessary foundation." Other writings, of what he terms Peacock's Cambridge school, such as those of De Morgan, Hankel states that he had not inspected; mention-

an introduction to the advanced theories of Gauss and Riemann; and for this purpose he went back to the unnoticed labours of Grassmann in Germany, to the writings of Peacock and De Morgan in England, and incidentally introduced into Germany the elaborate algebra of quaternions, invented and practised by Hamilton twenty years before that time. The papers of Riemann and Helmholtz similarly showed the necessity of a thorough investigation of the principles and foundations of ordinary or Euclidean geometry, and showed how consistent systems of geometry could be elaborated on other than Euclidean axioms. Only from that moment, in fact, did it become generally recognised that already, a generation before, two independent treatises on elementary geometry had been published in which the axiom of parallel lines was dispensed with and consistent geometrical systems developed. These were contained—as already stated —in the 'Kasan Messenger,' under date 1829 and

53. Non-Euclidean geometry.

ing only a short paper by Dr F. Gregory on Symbolical Algebra in the Edinburgh 'Transactions.' Whilst Hankel was delivering lectures on these fundamentals, Weierstrass in Berlin was likewise in the habit of introducing his lectures on the Theory of Analytic Functions by a discussion of the theory of Complex Numbers. This introduction was published, with Weierstrass's permission, in the year 1872 by Dr E. Kossak (in a programme of the Friedrichs-Werder Gymnasium), after lectures delivered by Weierstrass in 1865-66. To what extent Hankel may have been influenced by Weierstrass's lectures, which he seems to have attended after leaving Göttingen, is uncertain, for in spite of his very extensive references he does not mention Weierstrass. In Kossak's 'Elemente der Arithmetik' the term "permanence of formal rules" is not used, but the treatment of the extended arithmetic is carried on along the same lines—*i.e.*, not by an attempt to represent the complex quantities, but on the ground of maintaining the rules which govern the arithmetic of ordinary numbers. Great importance is also attached to the principle of inversion as having shown itself of value in the theory of elliptic functions, and being not less valuable in arithmetic. As stated above (p. 640, note), this principle is also insisted on by Peacock.

1830, the author being Lobatchevski; and in the appendix to an Introduction to Geometry, published by Wolfgang Bolyai at Maros Vasarheli, a town of Transylvania, the appendix being by the author's son, Johann Bolyai. The elder Bolyai having been a friend and correspondent of Gauss, and his speculations evidently of the same nature as those indicated by the latter in the above-mentioned correspondence, conjectures have been made as to which of the two originated the whole train of thought.[1] The independent investigations of Riemann and Helmholtz started from a differ-

[1] See above, p. 652, note. What is important from our point of view in the investigations of both Riemann and Helmholtz lies in the following points: First, Neither Riemann nor Helmholtz refers to the non-Euclidean geometry of Lobatchevski or Bolyai. This is not surprising in the case of Helmholtz, whose interest was originally not purely mathematical; in fact, we may incidentally remark how, in spite of his profound mathematical ability, he on various occasions came into close contact with mathematical researches of great originality and importance without recognising them—*e.g.*, the researches of Grassmann and Plücker. As regards Riemann, his paper was read before Gauss, who certainly knew all about Bolyai, and latterly also about Lobatchevski, of whom he thought so highly that he proposed him as a foreign member of the Göttingen Society. Gauss could therefore easily have pointed out to Riemann the relations of his speculations with his own and those of the other mathematicians named. Since the publication of the latest volume of Gauss's works, it has become evident that Gauss corresponded a good deal, and more than one would have supposed from reading Sartorius's obituary memoir, on the subject of non-Euclidean (astral or imaginary) geometry, notably with Gerling; and that several contemporary mathematicians, such as Schweikart, came very near to Gauss's own position. Second, although Riemann, and subsequently also Helmholtz, made use of the term "manifold" (*Mannigfaltigkeit*), it does not appear in the course of their discussion that they considered the space-manifold from any other than a metrical point of view. In fact, the manifold becomes in their treatment a magnitude (*Grösse*). It is true that Riemann does refer to certain geometrical relations not connected with magnitude but only with position, as being of great importance. These two points through which the researches of Riemann and Helmholtz stand in relation to other, and at the time isolated, researches, were dwelt on, the first by Beltrami, and the second by Cayley and Prof. Klein.

ent origin: both made use of the more general conception of an extended magnitude, introduced the notion of the curvature of space by analogy with Gauss's measure of curvature of a surface, and tried to express in algebraical formulæ the general and necessary properties of a magnitude which should form the foundation of a geometry. The relation of these algebraical results to those arrived at by the critical and purely geometrical methods of Lobatchevski and Bolyai were set out by Beltrami, who showed clearly that three geometries of two dimensions are possible—the Euclidean, that of Lobatchevski, where the three angles of a triangle are less than two right angles, and a third where they are more. He showed the analogy of the third with geometry on the sphere, and suggested the pseudo-sphere as a surface on which the second could be similarly represented. At the same time he indicated the generalisation through the algebraical formula of the conception of dimensions, and introduced the symbolical term geometry of four or more dimensions, as Grassmann and Cayley had done before him.[1] Through all these investigations a habit

[1] The geometry of non-Euclidean space, as well as the geometry of four or more dimensions (both usually comprised under the term "non-Euclidean geometry"), can now boast of an enormous literature, the enumeration of which alone would fill many pages. A complete bibliography up to the year 1878 is given in vols. i. and ii. of the American 'Journal of Mathematics' by Prof. Bruce Halsted, who has done much to make known to English readers the original writings of the pioneers in this subject. Later publications are referred to in Dr Victor Schlegel's papers ('Leopoldina,' xxii., 1886, Nos. 9-18): "Ueber Entwickelung und Stand der n-dimensionalen Geometrie," &c., &c. In France Houël published (beginning with the year 1866) translations of memoirs referring to this subject; in fact, he was almost the first to draw attention to this important modern departure. But it is almost exclusively owing to the various writings of Prof. Felix Klein that

has been introduced into mathematical writings which has not a little puzzled outsiders, and even exposed the logically rigorous deductions of mathematicians to the ridicule—not to say the contempt—of eminent philosophical authorities. The complete parallelism or correspondence of geometrical with algebraical notions —the possibility of expressing the former with perfect accuracy by the latter, and of retranslating the latter into the former, and this in more than one way, according to the choice of the space element (point, line, sphere), led to the habit of using purely geometrical presentable ideas as names for algebraical relations which had been generalised by the addition of more than a limited number of variables. Thus the conception of curvature, easily defined for a plane curve, and extended by Gauss to surfaces, was, by adding a third variable in the algebraic formula, applied to space. We are then told that it is necessary to understand what is meant by the curvature of space, this being a purely algebraical relation, not really presentable, but only formed by analogy from the geometrically presentable relations of geometry on a surface. In a similar

54. Curvature of space.

the different points of origin of this most recent mathematical speculation, which are to be found in the mathematical literature of all the principal nations, have been put in the true light and brought into connection. In fact, here, as in several other subjects, his publications, including his lithographed lectures on non-Euclidean geometry (delivered at Göttingen, 1893-94), serve as the best guide through the labyrinth and controversies of this intricate subject. See especially his article "Ueber die so-genannte nicht-Euclidische Geometrie" in vol. iv., 'Math. Ann.,' 1871. In this paper he connects the independent researches of Cayley (following Laguerre, 'Nouv. Ann. de Math.,' 1853), who in his sixth memoir on Quantics showed how metrical geometry can be included in projective geometry by referring figures to a fundamental fixed figure in space called by him the "Absolute," with the independent researches of Lobatchevski, Bolyai, Riemann, and Beltrami.

way the idea of the dimensions of space was extended, and four and more dimensions freely spoken of when really only a limited number is geometrically presentable. In the hands of mathematicians these terms are useful, and we may discard the criticism of philosophers and laymen as based on misunderstanding.[1] The introduction, however, into geometrical work of conceptions such as the infinite, the imaginary, and the relations of hyperspace, none of which can be directly imaged, has a psychological significance well worthy of examination.[2] It gives a deep insight into the resources and working of the mind. We arrive at the borderland of mathematics and philosophy.

[1] The most important philosophical criticism of the non-Euclidean geometry is that of Lotze, contained in the second book, chap. ii., of the 'Metaphysik' (1879, p. 249, &c.) It must not be forgotten that Lotze wrote at a time when the novel and startling conceptions put forward by popular writers on the subject had been employed in the interest of a spiritualistic philosophy, to the delusions of which some even of Lotze's friends had fallen a prey. This explains the severity of Lotze's criticisms, which are of the very same nature as those he pronounced many years earlier on similar aberrations (see 'Kleine Schriften,' vol. iii. p. 329). Those who are interested in following up the subject should refer to the writings of Friedr. Zöllner as collected in the four vols. of his 'Wissenschaftliche Abhandlungen' (Leipzig, 1878-81). They belong to the curiosities of the philosophical and scientific literature of that age, but can hardly claim a place in the history of thought.

[2] See the remark of Cayley in his Presidential Address ('Coll. Works,' vol. xi. p. 434): "The notion, which is really the fundamental one (and I cannot too strongly emphasise the assertion), underlying and pervading the whole of modern analysis and geometry, is that of imaginary magnitude in analysis and of imaginary space (or space as a *locus in quo* of imaginary points and figures) in geometry. I use in each case the word imaginary as including real. This has not been, so far as I am aware, a subject of philosophical discussion or inquiry. As regards the older metaphysical writers, this would be quite accounted for by saying that they knew nothing, and were not bound to know anything, about it; but at present, and considering the prominent position which the notion occupies—say even that the conclusion were that the notion belongs to mere technical mathematics or has reference to nonentities, in regard to which no science is possible—still it seems to me that (as a subject of philosophical discussion) the notion ought not to be thus ignored; it should at least be shown that there is a right to ignore it."

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There exists, moreover, an analogy between the manner in which these novel and extended ideas have been historically introduced and the mode of reasoning which led Sir W. R. Hamilton to the invention of a new and extended algebra—the algebra of quaternions. This analogy becomes evident if we study the small volume of Hermann Hankel, which appeared about the same time as Riemann's and Beltrami's fundamental geometrical dissertations.

55. Generalised conceptions.

The extension of Hamilton was only possible by dropping one of the fundamental principles of general arithmetic, the commutative principle of multiplication, which is symbolically expressed by saying that $a \times b$ is equal to $b \times a$. By assuming that $a \times b$ is equal to $- b \times a$, Hamilton founded a new general arithmetic on an apparently paradoxical principle. Similarly Lobatchevski and Bolyai constructed new geometries by dropping the axiom of parallel lines. Hankel made clear the significance of the new algebra, Riemann and Beltrami that of the new geometry. The practical performance anticipated and led up to the theoretical or philosophical exposition of the underlying principles. But there was a third instance in which a new science had been created by abandoning the conventional way of looking at things. This was the formation of a consistent body of geometrical teaching by disregarding the metrical properties and studying only the positional or projective properties, following Monge and Poncelet. The two great minds who worked out this geometry independently of the conception of number or measurement, giving a purely geometrical definition of distance and number, were Cayley in Eng-

land and Von Staudt in Germany. It was reserved for Prof. Felix Klein of Göttingen to show how the generalised notions of distance introduced into geometry by Cayley and Von Staudt opened out an understanding of the three geometries of Euclid, of Lobatchevski, and of Riemann.[1] We have to go back to the purely projective properties of space to understand these different possibilities. Lobatchevski attacked the problem practically, Riemann analytically, Klein geometrically. Through the labours of Klein the subject has arrived at a certain finality. And what was still wanting after he had written his celebrated memoir (which was approved and

56. Klein's exposition.

[1] See the note on p. 714, above; also 'Math. Ann.,' vol. iv. p. 573, and vol. vi. p. 112. Prof. Klein —following a usage in mathematical language — distinguishes three different geometries, the hyperbolic, the elliptic, and the parabolic geometry, corresponding to the possession by the straight line at infinity of two real or two imaginary (that is, none) or two coincident points. The whole matter turns upon the fact that, although metrical relations of figures are in general changed by projection, there is one metrical relation—known in geometry as the "anharmonic ratio" (in German *Doppelverhältniss*)—which in all projective transformations remains unchanged. As this anharmonic ratio of points or lines can be geometrically constructed without reference to measurement (Von Staudt, 'Geometrie der Lage,' 1847 and 1857), a method is thus found by which, starting from a purely descriptive property or relation, distance and angles—*i.e.*, metrical quantities—can be defined. Some doubts have been expressed whether, starting from the purely projective properties of space and building up geometry in this way (arriving at the metrical properties by the construction suggested by Von Staudt), the ordinary idea of distance and number is not tacitly introduced from the beginning. This may be of philosophical, but is not of mathematical, importance, as the main object in the mathematical treatment is to gain a starting-point from which the several possible consistent systems of geometry can be deduced and taken into view together. See on this point, *inter alia*, Cayley's remarks in the appendix to vol. ii. of 'Collected Works' (p. 604 *sqq.*), also Sir R. S. Ball's paper (quoted there), and more recently the discussion on the subject in Mr Bertrand Russell's 'Essay on the Foundations of Geometry' (1897, p. 31, &c.; p. 117, &c.). See also the same author's article on non-Euclidean Geometry in the supplement of the 'Ency. Brit.,' vol. xxviii.

commented on by Cayley) was later on supplied in consequence of a suggestion of his. The researches of Riemann, and still more those of Helmholtz, had not merely a mathematical, they had also a logical and a psychological, meaning. Space was conceived to be a threefold-extended manifold. There are other manifolds besides space—such, for instance, as the threefold-extended manifold of colours. Helmholtz came from the study of this manifold to that of space. Now the question arises as to the conditions or data which are necessary and sufficient for the foundations of a science like geometry. We have seen that the axiom of parallel lines is not required; we have also seen that the notion of distance and number can be generalised. What other data remain which cannot be dispensed with? Helmholtz had attempted to answer this question. But neither he nor Riemann had considered the possibility of a purely projective geometry. Now it is the merit of Prof. Klein to have seen that there exists a purely algebraical method by which this problem can be attacked. This is the method of groups referred to above, and 57. Sophus Lie.
applied by Sophus Lie to assemblages of continuously variable quantities. Klein was one of the first to recognise the power of this new instrument. He saw that the space problem was a problem of transformations, the possible motions in space forming a group with definite elements (the different freedoms of motion) which were continuously variable—*i.e.*, in infinitesimal quantities—and which returned into themselves under certain well-defined conditions. They possessed, moreover, in the maintenance of distance the algebraic property of in-

variance. He also expressed some doubt regarding the logical consistency of the assumptions of Helmholtz. Sophus Lie undertook this investigation, and thus brought the logical side of the labours of Riemann and Helmholtz to a final conclusion.[1] This is one of the celebrated instances where the rigorous algebraical methods have detected flaws in the more intuitional or purely geometrical process, and extended our knowledge of hidden possibilities.

But there is yet another branch of the great science of number, form, and interdependence, the principles and foundations of which had been handed down from earlier ages, where the critical and sifting process of the nineteenth century has led to an expansion and revolution of our fundamental ideas. Here also, as in so many other directions, the movement begins with Gauss. Hitherto I have spoken mainly of algebra or general arithmetic, of geometry, of the connections of both in the

[1] "Lie was early made aware by Klein and his 'program' that the space problem belonged to the theory of groups. . . . Ever since 1880 he had been pondering over these questions; he published his views first in 1886 on the occasion of the Berlin meeting of natural philosophers. Helmholtz's conception was itself unconsciously (but remarkably so, inasmuch as it dates from 1868) one belonging to the theory of groups, trying, as it did, to characterise the groups of the sixfold infinite motions in space, which led to the three geometries, in comparison with all other groups. He did this by fixing on the free mobility of rigid bodies—*i.e.*, on the existence of an invariant between two points as the only essential invariant. When Lie took up this problem in principle, as one belonging to the theory of groups, he recognised that for our space that part of the axiom of monodromy was unnecessary which added periodicity to the free mobility round a fixed axis. . . . The value of these investigations lies mainly in this, that they permit of our fixing for every kind of geometry the most appropriate system of axioms. . . . And they justly received in the year 1897 the first Lobatchevski prize awarded by the Society of Kasan" (M. Nöther, 'Math. Ann.,' vol. liii. p. 38). A lucid exposition of Lie's work will be found in Mr B. Russell's 'Essay,' &c., p. 47 *sqq.*

theory of forms and functions: there remains the science of numbers—of number in the abstract and also of the named numbers of ordinary arithmetic. Gauss's earliest labours were connected with this branch. Superseding the work of Fermat, Euler, and Legendre, he produced that great book with seven seals, the 'Disquisitiones Arithmeticæ.' The seals were only gradually broken. Lejeune Dirichlet did much in this way: others followed, notably Prof. Dedekind, who published the lectures of Dirichlet and added much of his own. The question may be asked, Have we gained any new ideas about numbers?

58. Theory of numbers.

In this abstract inquiry we can again facilitate our survey by distinguishing between the practical and the purely theoretical interests which stimulated it. Looking at the matter as well as the formal treatment by which it was rendered accessible, we may say Gauss not only taught us some very remarkable new properties of numbers—he also invented a new instrument or calculus for their investigation. Let us consider his work and that of his followers from these different points of view.

First, then, there were certain definite problems connected with the properties of numbers which had been handed down from antiquity. Such were the division of the circle into equal parts by a ready geometrical construction, the duplication of the cube, and the quadrature of the circle or the geometrical construction of the number π.[1] To the latter may be attached the

[1] See above, vol. i. p. 181, note. The student will find much interesting matter referring to these problems in Prof. Klein's little volume entitled 'Famous Problems in Elementary Geometry,' transl. by Beman and Smith, Boston and London, 1879. In it is also given

properties of the number e, the basis of the Napierian or natural logarithms, this number having been shown by Euler to stand in a remarkable arithmetical relation to the number π—a relation which could be very simply expressed if one had the courage to make use of the imaginary unit. As in the instance referred to above, when I dealt with the problem of the solution of the higher order of equations, so also in the case of the three celebrated problems now under review, the reasoning of the mathematicians of the nineteenth century lay largely in proving why these problems were insoluble or in defining those special cases in which they were soluble. Moreover, the labours of Gauss and the class of mathematicians who followed or read him were directed towards the defining and fixing of general conceptions, the study and elaboration of which embraced these single problems as special cases. Prime numbers had always been the object of special attention. Division and par-

an account of several mechanical contrivances for the solution of transcendental problems, or of those where the use of the compass and the ruler do not suffice. Although accurate constructions with a ruler and compass, or with either alone, were known to the ancients only in comparatively small numbers, approximations, and sometimes very close ones, seem to have been known. A very interesting example is Röber's construction of the regular heptagon, of which we read in the correspondence of Sir W. R. Hamilton with De Morgan (Life of Hamilton, by Graves, vol. iii. pp. 141, 534), and which was described by him in the 'Phil. Mag.,' February 1864. The approximation to the correctly calculated figure of the true septisection of the circle was so close that he could not discover, up to the 7th. decimal, whether the error was in the direction of more or less. On carrying the calculation further, he found the approximation to be such that a heptagon stepped round a circle equal in size to the equator would reach the starting-point within 50 feet. The inventor or discoverer of this method—Röber, an architect of Dresden—supposed that it was known to the ancient Egyptians, and in some form or other connected with the plans of the temple at Edfu, but on this point I have obtained no information. The question is not referred to in Prof. Cantor's 'History of Mathematics.'

tition of numbers had been studied, and many interesting formulæ had been found by induction, and subsequently proved—or not proved—by a multitude of ingenious devices. As in so many other directions of research so also here, the genius of Gauss gave a great impetus to progress by the invention of a definite calculus and an algorithm. This invention referred to the solution of what used to be known as indeterminate equations: to find two or more numbers—notably integers, which obey a certain algebraical relation. For one large class of these problems (which already occupied the ancient geometers), viz., those of the divisibility of one number by another (called the modulus) with or without residue, Gauss invented the conception and notation of a congruence. Two numbers are congruent if when divided by a certain number they leave the same remainder. "It will be seen," says Henry Smith, "that the definition of a congruence involves only one of the most elementary arithmetical conceptions—that of the divisibility of one number by another. But it expresses that conception in a form so suggestive of analysis, so easily available in calculation and so fertile in new results, that its introduction into arithmetic has proved a most important contribution to the progress of the science."[1] Notably the analogy with ordinary algebraic equations and the possibility of transferring the properties and treatment of these was at once evident. It became a subject of

59. Gauss's theory of congruences.

[1] See Henry J. S. Smith in his most valuable 'Report on the Theory of Numbers' (Brit. Assoc., 1859-65, six parts. Reprinted in 'Collected Math. Papers,' vol. i. pp. 38-364). It gives a very lucid account of the history of this department of mathematical science up to the year 1863.

interest to determine the residues of the powers of numbers. A number is said to be a quadratic, cubic, or biquadratic residue of another (prime) number (the modulus) if it is possible to find a square, cube, or biquadratic number which is congruent with the first number. The theory of congruences was a new calculus: as such it was, like the theory of determinants or of invariants or the general theory of forms, a tactical device for bringing order and simplicity into a vast region of very complicated relations. Gauss himself wrote about it late in life to Schumacher.[1] "In general the position as regards all such new calculi is this—that one cannot attain by them anything that could not be done without them: the advantage, however, is, that if such a calculus corresponds to the innermost nature of frequent wants, every one who assimilates it thoroughly is able—without the unconscious inspiration of genius which no one can command—to solve the respective problems, yes, even to solve them mechanically in complicated cases where genius itself becomes impotent. So it is with the invention of algebra generally, so with the differential calculus, so also—though in more restricted regions—with Lagrange's calculus of variations, with my calculus of congruences, and with Möbius's calculus. Through such conceptions countless problems which otherwise would remain isolated and require every time (larger or smaller) efforts of inventive genius, are, as it were, united into an organic whole." But a new calculus frequently does more than this. In the course of its

[1] See 'Briefwechsel,' &c., vol. iv. p. 147; also Gauss's 'Werke,' vol. viii. p. 298.

application it may lead to a widening of ideas, to an enlargement of views, to a removing of artificial and conventional barriers of thought. As I stated early in this chapter, the attempts of Gauss to prove the fundamental theorem of algebra, that every equation has a root, suggested to him the necessity of introducing complex numbers; the development of the theory of congruences and of residues—notably of the higher residues—confirmed this necessity. In the year 1831, in his memoir on biquadratic residues, he announces it as a matter of fundamental importance. In the earlier memoir he had treated this extension of the field of higher arithmetic as possible, but had reserved the full exposition. And before he redeemed this promise the necessity of doing so had been proved by Abel and Jacobi, who had created the theory of elliptic functions, showing that the conception of a periodic function (such as the circular or harmonic function) could be usefully extended into that theory, if a double period—a real and an imaginary one—were introduced. A simplification similar to that which this bold step led to in the symbolic representation of those higher transcendents, had been discovered by Gauss to exist in the symbolical representation of the theory of biquadratic residues which only by the simultaneous use of the imaginary and the real unit "presented itself in its true simplicity and beauty." In this theory it was necessary to introduce not only a positive and negative, but likewise a lateral system of counting—*i.e.*, to count not only in a line backwards and forwards, but also sideways in two directions, as Gauss showed very plainly in the now familiar manner. At the

60. Generalised conception of number.

same time a metaphysical question presented itself—viz., Can such an extension into more than two dimensions be consistently and profitably carried out? Gauss had satisfied himself that it could not;[1] but the proof of this was only given in more recent times by Weierstrass, who definitely founded the whole discussion of the subject on the logical principle "that the legitimacy of introducing a number into arithmetic depends solely on the definition of such number." And this leads me to another extension in the region of number suggested by Gauss's treatment, which has also become fundamental, and, in the hands of Dirichlet, Kummer, Liouville, Dedekind, and others, has remodelled the entire science of higher arithmetic. It is based on the logical process of the

[1] A concise history of this subject is given by Kossak in the Program referred to above, p. 712, note. Gauss had promised to answer the question, "Why the relations between things which have a manifoldness of more than two dimensions would not admit of other" (than the ordinary complex numbers introduced by him) "fundamental quantities being introduced into general arithmetic?" He never redeemed his promise. In consequence of this, several eminent mathematicians, notably Hankel, Weierstrass, and Prof. Dedekind, have attempted to reply to this question, and to establish the correctness of the implied thesis according to which any system of higher complex numbers becomes superfluous and useless. Prof. Stolz, in the first chapter of the second volume of his 'Allgemeine Arithmetik,' gives an account of these several views, which do not exactly coincide. In general, however, the proof given by Weierstrass, and first published by Kossak, has been adopted. This proof is based upon the condition that the product of several factors cannot disappear except one of its factors is equal to zero. "We must, therefore, exclude from general arithmetic complex numbers consisting of three fundamental elements. This is, however, not necessary if the use of them be limited" by some special conditions (Kossak, *loc. cit.*, p. 27). In the course of the further development of this matter Weierstrass arrives at the fundamental thesis "that the domain of the elementary operations in arithmetic is exhausted by addition and multiplication, including the inverse operations of subtraction and division." "There are," says Weierstrass, "no other fundamental operations—at least it is certain that no example is known in analysis where, if an analytical connection exists at all, this cannot be analysed into and reduced to those elementary operations" (p. 29).

inversion of operations in the most general manner. In the direct process we build up algebraical formulæ—called equations or forms—by a combination of addition and multiplication. We can omit subtraction and division, as through the use of negative quantities and fractions these are reduced to the former. Now, given the most general algebraical equation or form, we can search out and define the simple factors or forms into which it can be split up, and these factors and their products we can take to serve as the definition of numbers. The question then arises, What are the properties of numbers thus inversely defined? and, secondly, Do these numbers exhaust or cover the whole extent of number as it is defined by the uses of practical life? The answer to the former question led to the introduction of complex and subsequently of ideal numbers; the discovery by Liouville that the latter is not the case has led to the conception of transcendental, *i.e.*, non-algebraic, numbers.

61. Process of inversion.

The idea of generalising the conception of number, by arguing backward from the most general forms into which ordinary numbers can be cast by the processes of addition and multiplication, has led to a generalised theory of numbers. Here, again, the principal object is the question of the divisibility of such generalised algebraical numbers and the generalised notion of prime numbers—*i.e.*, of prime factors into which such numbers can be divided. Before the general theory was attempted by Prof. Dedekind, Kronecker, and others, the necessity of some extension in this direction had already been discovered by the late Prof. Kummer of

62. Kummer's ideal numbers.

Berlin when dealing with a special problem. This was no other than the celebrated problem of the division of the circle into equal parts, which had been reduced by Gauss to an arithmetical question. Gauss had shown that the accurate geometrical solution of this problem depended on the solution of certain simple binomial forms or equations. The study of such forms accordingly became of special interest: it necessitated the employment of the extended notion of number called by Gauss that of complex numbers. Now it is one of the fundamental laws in the theory of ordinary numbers that every integer can be divided only in one way into prime numbers. This law was found to break down at a certain point if complex numbers were admitted. Kummer, however, suggested that the anomaly disappeared if we introduced along with the numbers he was dealing with other numbers, which he termed ideal numbers—*i.e.*, if we considered these complex factors to be divisible into other prime factors. The law of divisibility was thus again restored to its supreme position. These abstract researches led to the introduction of a very useful conception—the conception not only of generalised numbers, but also of a system (body, corpus, or region) of numbers;[1] comprising all numbers which, by the

[1] The idea of a closed system or domain of generalised numbers has revolutionised the theory of numbers. Originally the theory of numbers meant only the theory of the common integers, excluding complex numbers. Gauss, in the introduction to the 'Disquisitiones,' limits the doctrine in this way. He excludes also the arithmetical theories which are implied in cyclotomy—*i.e.*, the theory of the division of the circle; stating at the same time that the principles of the latter depend on theories of higher arithmetic. This connection of algebraical problems with the theory of numbers became still more evident in the labours of Gauss's successors—Jacobi and Lejeune Dirichlet, and was surprising to them. "The

ordinary operations of arithmetic, can be formed out of the units or elements we start with. Thus all rational integers form a system; we can compound them, but also resolve them into their elements. Where we introduce new elements or units we only arrive at correct laws if we are careful to cover the whole field or system which is measured by the application of the fundamental operations of arithmetic. Throughout all our abstract reasoning it is the fundamental operations which remain permanent and unaltered,—a rule which

reason for this connection is now completely cleared up. The theory of algebraical numbers and Galois's 'theory of equations' have their common root in the general theory of algebraical systems; especially the theory of the system of algebraical numbers has become at the same time the most important province of the theory of numbers. The merit of having laid down the first beginnings of this theory belongs again to Gauss. He introduced complex numbers, he formulated and solved the problem of transferring the theorems of the ordinary theory of numbers, above all, the properties of divisibility and the relation of congruence, to these complex numbers. Through the systematic and general development of this idea,—based upon the far-reaching ideas of Kummer,—Dedekind and Kronecker succeeded in establishing the modern theory of the system of algebraical numbers" (Prof. Hilbert in the preface to his "Theorie der Algebraischen Zahlkörper," 'Bericht der Math. Ver.,' vol. iv. p. 3). In the further course of his remarks Prof. Hilbert refers to the intimate connection in which this general or analytical theory of numbers stands with other regions of modern mathematical science, notably the theory of functions. "We thus see," he says, "how arithmetic, the queen of mathematical science, has conquered large domains and has assumed the leadership. That this was not done earlier and more completely, seems to me to depend on the fact that the theory of numbers has only in quite recent times arrived at maturity." He mentions the spasmodic character which even under the hands of Gauss the progress of the science exhibited, and says that this was characteristic of the infancy of the science, which has only in recent times entered on a certain and continuous development through the systematic construction of the theory in question. This systematic treatment was given for the first time in the last supplement to Dedekind's edition of Dirichlet's lectures (1894, 4th ed., p. 134). A very clear account will also be found in Prof. H. Weber's 'Lehrbuch der Algebra' (vol. ii., 1896, p. 487, &c.) He refers (p. 494) to the different treatment which the subject has received at the hands of its two principal representatives — Prof. Dedekind (1871 onwards) and Kronecker (1882) — and tries to show the connection of the two methods.

as we saw above, was vaguely foreshadowed by Peacock, and expressly placed at the head of all mathematical reasoning by Hermann Hankel. In passing it may also be observed how the notion of a system of algebraical numbers, which belong together as generated in certain defined ways, prepares us for the introduction of that general theory of groups which is destined to bring order and unity into a very large section of scattered mathematical reasoning. The great importance of this aspect is clearly and comprehensively brought out in Prof. H. Weber's Algebra. Nothing could better convince us of the great change which has come over mathematical thought in the latter half of the nineteenth century than a comparison of Prof. Weber's Algebra with standard works on this subject published a generation earlier.

63. Modern algebra.

I have shown how the definition of algebraical numbers has led to an extension and generalisation of the conception of number. Another question simultaneously presented itself, Does this extension cover the whole field of numbers as we practically use them in ordinary life? The reply is in the negative. Practice is richer than theory. Nor is it difficult to assign the reason of this. Numbering is a process carried on in practical life for two distinct purposes, which we distinguish by the terms counting and measuring. Numbering must be made subservient to the purpose of measuring. Thus difficulties arising out of this use of numbers for measuring purposes presented themselves early in the development of geometry in what are called the incommensurable quantities: taking the side of a square as ten, what is the number which measures the

64. Algebraical and transcendental numbers.

diagonal? Assume that we prolong the side of the square indefinitely, we have a clear conception of the position of the numbers 15, 20, 30, &c.; but what is the exact number corresponding to the length of the diagonal? This led to the invention of irrational numbers: it became evident that by introducing the square root of the number 2 we could accurately express the desired number by an algebraical operation. But there are other definite measurements in practical geometry which do not present themselves in the form of straight lines, such as the circumference of a circle with a given radius. Can they, like irrational quantities, be expressed by definite algebraical operations? Practice had early invented methods for finding such numbers by enclosing them within narrower and narrower limits; and an arithmetical algorithm, the decimal fraction, was invented which expressed the process in a compact and easily intelligible form. Among these decimal fractions there were those which were infinite—the first instances of infinite series—progressing by a clearly defined rule of succession of terms: others there were which did not show a rule of succession that could be easily grasped. Much time was spent in devising methods for calculating and writing down, *e.g.*, the decimals of the numbers π and e.[1]

It will be seen from this very cursory reference to the practical elements of mathematical thought how the ideas or mental factors which we deal with and

[1] The transcendent nature of the numbers e and π was first proved by Hermite and Prof. Lindemann. The proofs have been gradually simplified. A lucid statement will be found in Klein's 'Famous Problems,' p. 49 *sqq.*

string together in mathematical reasoning are derived from various and heterogeneous sources. We begin

65. Counting and measuring.

with counting, then we introduce measuring; in both cases we have definite elements or units which may serve to express order or quantity or both, and we have definite conventional operations; then we have symbols which may denote order or quantity or operation. With these devices we perform on paper certain changes, and we get accustomed to use indiscriminately these heterogeneous conceptions, arithmetical, geometrical, algebraical—nay, even dynamical, as when Newton introduced the conception of a flow or fluxion. As mathematics is an instrument for the purpose of solving practical problems, skill in alternately and promiscuously using these incongruous methods goes a very long way. Geometrical, mechanical evidence helps frequently where pure logic comes to a standstill, and pure logic must help and correct where apparent evidence might deceive us. Mathematics and science generally have always progressed by this alternate use of heterogeneous devices, and will probably always do so. The straight line of pure logic has but very meagre resources, and resourcefulness is the soul of all progress. But though this may be so in practice, there are two other interests which govern scientific reasoning. There is the love of consistency and accuracy, and of clean and transparent, as distinguished from muddled and scamped, work. The latter leads inevitably into serious errors and paradoxes, as the great mathematicians, Gauss, Cauchy, Abel, pointed out early in the century. Mathematics then frequently

exhibited the slovenliness of a man who talks at the same time in more than one language, because he is too negligent to arrange his thoughts clearly. Then there come in the demands of the teacher who has to introduce abstract and difficult subjects in a clear, consistent, and simple manner, taking heed that with the elements he does not introduce the sources of future error. The same interest that led in ancient times to the composition of the Elements of Euclid has led, in the higher education of the nineteenth century, beginning with the École Polytechnique and ending with Weierstrass's famous courses of lectures at Berlin, to a revision and recasting of the whole elementary framework of mathematics. In the mean time the resourcefulness in applied mathematical thought which ever since the age of Newton has characterised the individual research of this country, has opened out new vistas and afforded much material for critical siftings and strict definitions. Both qualities were united in the great mind of Gauss with a regrettable absence of the love of teaching and the communicative faculty. Like Newton's 'Principia,' his greatest works will always remain great storehouses of thought; while his unpublished remains might be compared to the Queries appended to the 'Opticks' and to the 'Portsmouth Papers.'

Several eminent mathematicians in France, Germany, and Italy have been for many years[1] working at the

[1] The literature of this subject has been rapidly increasing since the year 1872, — the approximate date of the following publications, which created an epoch: R. Dedekind, 'Stetigkeit und irrationale Zahlen' (Braunschweig, 1872); E. Heine, "Die

clearer enunciation of the fundamental conceptions of the science, and though the ways in which they approach the subject are different, a general consensus seems to be within view as to the elementary definitions. The main difficulty lies in the introduction into pure arithmetic of the ideas which are forced upon us when

Elemente der Functionenlehre" ('Journal für Mathematik,' vol. lxxiv. p. 172, 1872). This paper refers both to Weierstrass's and Cantor's theories; H. Kossak, in the pamphlet referred to above (p. 712, note). This contains the principles of Weierstrass's theory; C. H. Méray, 'Nouveau Précis d'Analyse infinitésimale' (Paris, 1872). The first comprehensive publication of Georg Cantor belongs to the year 1883, 'Grundlagen einer allgemeinen Mannigfaltigkeitslehre' (Leipzig, Teubner). It was preceded by various articles in the 'Journal für Mathematik,' vol. lxxvii. p. 257, vol. lxxxiv. p. 82, and 'Math. Ann.,' vol. xv. p. 1, in which he had introduced and defined several of the terms and conceptions that have since become generally accepted in writings on this subject. These earlier publications, by—or referring to—the pioneers in this new province of mathematical thought, were followed by a number of further expositions by Cantor, Dedekind, and Weierstrass. The principal writings of Cantor have been republished in the 'Acta Mathematica,' vol. ii. Prof. Dedekind published in the year 1888 an important pamphlet, 'Was sind und was sollen die Zahlen,' and has incorporated many of the results of his researches in his later editions of Dirichlet's 'Lectures'; whilst the lines of reasoning peculiar to Weierstrass have become better known through the writings of his pupils and the collected edition of his mathematical works which is now in progress. A complete bibliography is given in three important articles in vol. i. of the German 'Math. Encyc.' by Profs. Schubert (p. 1, &c.), Pringsheim (p. 48, &c.), and Schönflies (p. 184, &c.). Important works, giving a summary and analysis of these various researches, now exist in the mathematical and philosophical literature of France, Germany, Italy, and England. Like the non-Euclidean geometry, the subject has attracted considerable attention also outside purely mathematical circles. Notably Cantor's writings have been exhaustively dealt with from a philosophical point of view — in Germany by Walter Brix (Wundt's 'Philosophische Studien,' vol. v. p. 632, vol. vi. p. 104 and 261), and by B. Kerry, 'System einer Theorie der Grenz-begriffe' (Leipzig und Wien, 1890); in France by M. Louis Couturat, 'De l'Infini mathématique' (Paris, 1896); and latterly in this country by Mr Bertrand Russell, 'The Principles of Mathematics,' vol. i. (Cambridge, 1903). Italian mathematicians have also dealt largely with the subject, notably G. Peano, who published an important work, 'Arithmetices principia nova methodo exposita' (Turin, 1889).

we apply the counting process to the needs of geometry and physics. We are here confronted with notions which require to be arithmetically defined—the infinite and the continuous. The same notions at the beginning of the century attracted the attention of eminent analysts like Cauchy. It is now clear, thanks to the labours of Prof. Georg Cantor of Halle, that for mathematical purposes we must distinguish between the indefinitely great and the actually infinite in the sense of the transfinite. To deal with the actually infinite, as distinguished from the immeasurably or indefinitely great, we have to introduce new notions and a new vocabulary. For instance, in dealing with infinite aggregates, the proposition that the part is always less than the whole is not true. Infinities, indeed, differ, but not according to the idea of greater and smaller, of more or less, but according to their order, grade, or power (in German *Mächtigkeit*). Two infinities are equal, or of the same power, if we can bring them into a one-to-one correspondence. Prof. Cantor has shown that the extended range of numbers termed algebraic have the same power as the series of ordinary integers—one, two, three, &c.—because we can establish a one-to-one correspondence between the two series—*i.e.*, we can count them. He has further shown that if we suppose all numbers arranged in a straight line, then in any portion of this line, however small, there is an infinite number of points which do not belong to a countable or enumerable multitude. Thus the continuum of numerical values is not countable—it belongs to a different

66.
Georg Cantor's theory of the transfinite.

grade of infinity; it has a higher, perhaps the second, power.[1]

In all these, and in many similar investigations, a conception has gradually emerged which was foreign to older mathematics, but which plays a great and useful part in modern mathematical thought. Older mathematics, ever since the introduction of general arithmetic or algebra, centred in the conception of equality and in the solution of equations. Everything was reduced to magnitude. But there are other relations besides those of magnitude, of more or less. Often in practical pursuits, if we cannot find a counterpart or write down an exact numerical equation, we can gain information

67. Correspondence.

by a correspondence. This conception of correspondence plays a great part in modern mathematics. It is the fundamental notion in the science of order as distinguished from the science of magnitude. If older mathematics were mostly dominated by the needs of mensuration, modern mathematics are dominated by the conception of order and arrangement. It may be that this tendency of thought or direction of reasoning goes hand in hand with the modern discovery in physics, that the changes in nature depend not only or not so much on the quantity of mass and energy as on their distribution or arrangement.

With these reflections we touch the limits of mathe-

[1] A summary of Prof. Cantor's work is given by Prof. Schönflies in the 'Encyklop. Math. Wiss.,' vol. i. p. 184 *sqq.* The importance of accurate definitions and distinctions regarding the infinite and the continuous is dwelt on and the different recent theories set forth in a very lucid address to the London Math. Society by Prof. Hobson, "On the Infinite and Infinitesimal in Mathematical Analysis," November 1902.

matical thought and enter the region of metaphysics. Like other lines of reasoning which have occupied us in former chapters, the exact and rigid definitions and deductions of arithmetic and geometry lead us up to that other large department of our subject—philosophic thought. Many eminent mathematicians of recent years have noticed this tendency, and have urged the mutual help which arithmetic and geometry on this side, logic and psychology on that, may derive from each other. The names of Helmholtz, Georg Cantor, and Dedekind in Germany; of M. Tannery and M. Poincaré in France; of Peano and Veronese in Italy, stand prominently forward abroad; while England can boast of having cultivated, much earlier, by the hands of De Morgan and Boole, a portion at least of this borderland, and of having in recent years taken up the subject again in an original and independent manner.[1] Cayley, in his address to the British Association in 1883, has said: "Mathematics connect themselves on the one side with common life and the physical sciences; on the other

[1] I refer to the important but unfinished works of Mr Whitehead on 'Universal Algebra' (vol. i., 1898), and of Mr Bertrand Russell on 'The Principles of Mathematics' (vol. i., 1903). I must defer a more detailed appreciation of these and other writings of this class, such as those of the late Prof. Ernst Schröder ('Algebra der Logik,' 3 vols., 1890-95) and of Prof. Gottlob Frege (see an account of his writings in the appendix to Mr Russell's 'Principles'). They belong largely to a department of philosophical thought which may be termed "the Philosophy of the Exact Sciences." This deals with two great questions—the logical foundations of scientific reasoning, and the general outcome and importance of scientific thought, not for technical purposes, but in the great edifice of human thought which we may term Philosophy. It deals with what has been called "the Creed of Science" and its value. Stanley Jevons and Prof. Karl Pearson in this country, Prof. Mach in Germany, and M. Poincaré in France, have treated the philosophy of science in one or both of these aspects.

side with philosophy in regard to our notions of space and time, and in the questions which have arisen as to the universality and necessity of the truths of mathematics and the foundation of our knowledge of them"; and he subsequently refers specially to the "notion which is really the fundamental one underlying and pervading the whole of modern analysis and geometry," meaning the complex magnitude, as deserving to be specially discussed by philosophers. Beginnings of the philosophical treatment of this and other questions indeed exist. The questions are still *sub judice*, and the historian can merely refer to their existence and importance.

There is, however, one controversy which has arisen out of these and similar speculations, and out of the desire to bring unity and consistency into the fundamental notions of elementary as well as higher mathematics, which deserves to be specially mentioned, because it occupies a prominent place in foreign literature, having given rise to a special term, and thus commanding more general attention. Prof. Klein of Göttingen, under whose master-hand many abstract and obscure subjects have become plain and transparent, has prominently brought the subject before the scientific public in a recent address.[1] I refer to the tendency represented in its extreme form by the late Prof. Kronecker of Berlin, to reduce all mathematical conceptions to the fundamental arithmetical operations with integral numbers, banishing not only all geometrical and dynamical conceptions, such as those of continuity and flow, but

68. Arithmetising tendency in mathematics.

[1] 'Ueber Arithmetisirung der Mathematik' (Göttingen, 1895).

also such apparently algebraical notions as those of irrational and complex quantities. This attempt is an outcome of the school of Weierstrass, which has done so much to banish vagueness and introduce precision into modern text-books.

Opposed to this so-called arithmetising[1] tendency is the equally emphatic view, strongly urged by the late Prof. Paul Du Bois-Reymond in his general theory of Functions, that the separation of the operations of counting and measuring is impossible, and, if it were possible (as, since the publication of his work, the fuller expositions of Kronecker and his followers have tried to show that it is), would degrade mathematics to a mere play with symbols.[2] He tries to show that such is philosophically impossible, and finds a support for his view in the historical genesis of the idea of irrational numbers in the incommensurable magnitudes of Euclid and ancient geometry. Prof. Klein in his address favours the arithmetical tendency as destined to introduce logical

[1] The term seems to have been coined by Kronecker. See Prof. Pringsheim in the 'Encyklop. Math. Wiss.,' vol. i. p. 58, note 40. Kronecker's position is set forth in 'Journal für Math.,' vol. ci. pp. 337-355, 1887.

[2] "The separation of the conception of number and of the analytical symbols from the conception of magnitude would reduce analysis to a mere formal and literal skeleton. It would degrade this science, which in truth is a natural science, although it only admits the most general properties of what we perceive into the domain of its researches ultimately to the rank of a mere play with symbols, wherein arbitrary meanings would be attached to the signs as if they were the figures on the chessboard or on playing-cards. However amusing such a play might be, nay, however useful for analytical purposes the solution would be of the problem,—to follow up the rules of the signs which emanated from the conception of magnitude into their last formal consequences,—such a literal mathematics would soon exhaust itself in fruitless efforts; whereas the science which Gauss called with so much truth the science of magnitude possesses an inexhaustible source of new material in the ever-increasing field of actual perceptions," &c., &c. ('Allgemeine Functionen-Theorie,' 1882, p. 54.)

precision and consistency into the foundations of mathematics, and everywhere to further the very necessary process of critical sifting; but he denies that pure logic can do all, and points to the valuable assistance and suggestive power of geometrical construction and representation.[1] Most of my readers will no doubt agree with this view. Indeed the perusal of the foregoing chapters must have produced on their minds the conviction that, so far as the advance of science and also of mathematics is concerned, it largely depends upon the introduction of different aspects leading to different courses of reasoning. The unification of all of these into one consistent and uncontradictory scheme, though it remains a pious hope and far-off ideal, has not been the prominent work of the nineteenth century. Rather, wherever it has been attempted it has had a narrowing effect, and has resulted in a distinct curtailment of the great and increasing resources of Scientific Thought.

[1] Prof. Klein summarises the opinion which he holds as to the present task of mathematical science as follows: "Whilst I everywhere demand the fullest logical elaboration, I at the same time emphasise that *pari passu* with it the intuitive representation of the subject should be furthered in every possible manner. Mathematical developments which have their origin in intuition cannot count as a firm possession of science unless they have been reduced to a strict logical form. On the other side, the abstract statement of logical relations cannot satisfy us until their importance for every form of representation has been clearly demonstrated, so that we recognise the manifold connections in which the logical scheme stands to other departments of knowledge according to the field of application which we select. I compare mathematical science to a tree which stretches its roots ever deeper into the soil, and at the same time expands its branches freely upwards. Are we to consider the root or the branches as the more important part? The botanist will tell us that the question is wrongly put, and that the life of an organism consists in the interaction of its various parts" (*loc. cit.*, p. 91).